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Exceptional Seifert Fibered Surgeries on Montesinos Knots and Distinguishing Smoothly and Topologically Doubly Slice Knots

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Exceptional Seifert Fibered Surgeries on Montesinos Knots and Distinguishing Smoothly and Topologically Doubly Slice Knots

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To my wife, Sarah.

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The results presented in this thesis pertain to two distinct areas of low-dimensional topology. First, we give a classification of small Seifert fibered surgeries on hyperbolic pretzel knots, as well as a near-classification of small Seifert fibered surgeries on hyperbolic Montesinos knots. Along with recent results of Ichihara-Masai [IM13], these results complete the classification of all exceptional Dehn surgeries on arborescent knots.

Second, we exhibit an infinite family of smoothly slice knots that are topologically doubly slice, but not smoothly doubly slice. A subfamily of these knots is then used to show that the subgroup of the smooth double concordance group consisting of topologically doubly slice knots is infinitely generated. One corollary of these results is that there exist infinitely many rational homology 3–spheres (with nontrivial first homology) that embed topologically, but not smoothly, into the 4–sphere.

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Chapter 1

Introduction and Overview

1.1 Motivation and Introduction

The subject of concern in the present thesis is the field of low-dimensional topology, the goal of which is to understand 3–dimensional and 4–dimensional manifolds, as well as the interaction between them. An n–dimensional manifold is a topological space that, in a small neighborhood of each point in the space, is indistinguishable from the familiar Euclidean space of dimension n: $\mathbb{R}^n$. For example, a 1–manifold looks locally like a line, and a 2–manifold, also called a surface, looks locally like a plane. Thus, a 3–manifold is a space that looks locally like the spacial world we inhabit, and a 4–manifold allows for a fourth dimension at each point. From this view point, the objects are highly intuitive objects and are well-deserving of rigorous mathematical study.

The theme of the material that follows is the interaction between 1–, 2–, 3–, and 4–manifolds. It turns out that these objects are very complicated mathematically, and many years of fruitful research has revealed only some of the many interesting aspects of their structure and interaction. One way to understand complicated objects such as 3–manifolds and 4–manifolds is to study them through the lens of knot theory. A circle is a simple example of a

1–manifold, and knot theory is the study of "the placement problem" for the circle. In other words, a knot is a placement of the circle in 3–space, and knot theory is the study of all possible such placements, called embeddings.

The present work is interested in two aspects of knot theory. The first is called Dehn surgery and is a highly useful technique for studying 3–manifolds. Dehn surgery is the process of taking a knot in 3–space, and removing a tubular neighborhood of the knot only to glue it back in in a nontrivial way. This process can produce nontrivial 3–manifolds whose properties can be inferred, in many cases, from properties of the original knot. In this way, the study of a complicated object, such as a 3–manifold, is reduced to a more intuitive, but still mathematically rich study, namely, that of knots in 3–space. This is the subject of study in Chapter 2.

The second aspect of knot theory discussed below is the study of slice knots and doubly slice knots. Just as we can study knots in 3–space when considering the "placement problem" for the circle, we can also study objects called 2–knots in 4–space. These are embeddings of the 2–dimensional sphere into 4–space, and the study of such objects might be called the "placement problem" for spheres. Give such a 2–knot in 4–space, we can consider its intersection with the standard copy of 3–space that divides 4–space. This intersection sometimes turns out to be a knot in 3–space. A knot that arises in this way is called a slice knot, because it is a slice of a 2–knot. If the original 2–knot is trivially embedded in 4–space, i.e., if it is unknotted, then any knot arising as the sliced cross-section is called doubly slice.

Based on the set-up discussed in the previous paragraph, it is not hard to believe that the study of slice knots and doubly slice knots is intimately related to the study of 4–dimensional manifolds. So, once again, we have reduced the study of complicated objects to the more intuitive realm of knot theory. It is a beautiful fact that much of the complicated behavior of 4–manifolds, such as the difference between smooth and topological 4–manifolds, can be realized by objects as simple as knots. This is the subject of study in Chapter 3.

1.2 Introduction to exceptional Dehn surgery and summary of results

Dehn surgery provides a natural and powerful way of connecting the study of 3–manifolds to the study of knots. In light of Perelman's resolution of Thurston's Geometrization Conjecture [Per02, Per03a, Per03b, Thu82a], exceptional Dehn surgery gives us an excellent framework to understand 3–manifolds. In particular, we know that most knots are hyperbolic and that performing Dehn surgery on a hyperbolic knot almost always gives rise to a hyperbolic manifold [Thu82b]. Because of this, it is natural to try to understand when a hyperbolic knot admits a non-hyperbolic Dehn surgery. Such a surgery is called *exceptional*. Much work has been done to try to understand this phenomenon, but there remain many interesting open questions and conjectures (for an excellent survey, see [Gor09a]).

One type of exceptional Dehn surgery, those producing small Seifert fibered spaces, has evaded our understanding even more than the others. This area is the focus of the first half of the present work. The results from Chapter 2 also appear in [Mei14], and provide the final piece in the classification of exceptional Seifert fibered surgeries on pretzel knots, as well as a result that, along with recent work of Ichihara and Masai [IM13], completes the classification for all Montesinos knots. In [IM13], it was shown that $K[-1/2, 2/5, 1/(2q+1)]$ with $q \geq 5$ admits no Seifert fibered surgeries. The main results of Chapter 2 can be stated as follows.

Theorem A ([Mei14])**.** Let $K_q = P(-2, 3, 2q+1)$ for $|q| \geq 3$, and let $K^a = P(3, -3, a)$ for $a \in [2, 6]$.

- The only hyperbolic pretzel knots admitting Seifert fibered surgeries are the K_q and K^a.
- The K_q each admit exactly two small Seifert fibered surgeries when $q \geq 4$, while the K^a each admit exactly one.

Theorem B ([Mei14])**.** If K is a hyperbolic, non-pretzel Montesinos knot admitting a Seifert fibered surgery, then one of the following holds:

- $K = K[1/3, -1/3, 2/5]$, and K admits one Seifert fibered surgery.
- $K = K[-1/2, 1/3, 2/(2a+1)]$ with $a = 3, 4,$ or 5, and K admits three Seifert fibered surgeries if $a = 3$ or 4, and two if $a = 5$.

- $K = K[-1/2, 2/5, 1/(2q+1)]$ with $q = 1, 2$, or 3, and K admits three Seifert fibered surgeries if $q = 1$, two if $q = 2$, and one if $q = 3$.

- $K = K[-1/2, 2/5, 1/(2q+1)]$ and $q \geq 5$.

As a corollary, we answer the Seifert Fibered Space Conjecture for all Montesinos knots, i.e., any exceptional Seifert fibered surgery on a hyperbolic Montesinos knot must be integral. Portions of Theorem A and B were independently proved by Wu (see [Wu12b, Wu13]) using the computer program *Snappex*.

Note that the surgeries in Theorem A were previously known, and $K_3 = P(-2, 3, 7)$ is known to admit precisely three finite Seifert fibered space surgeries, including two lens space surgeries [EM02].

1.3 Introduction to doubly slice knots and summary of results

Knots don't just give us information about 3–dimensional topology, they also provide a means of studying 4–dimensional phenomena. One particularly interesting instance of this is the study of slice knots and doubly slice knots. A knot is called *slice* if it can be realized as the equator of an embedding of S^2 into S^4. Similarly, a knot is called *doubly slice* if it can be realized as the equator of an unknotted embedding of S^2 in S^4. Slice knots have been used to study exceptional 4–manifolds (see [GS99] for an overview), and the study of doubly slice knots is related to the study of 3–manifolds that embed in S^4 (see, for example, [Don12, GL83]). In these ways, by understanding when knots are slice or doubly slice, we are capturing some of the subtlety and profundity of 4–dimensional topology in the more approachable study of knots in S^3.

Even though slice knots and doubly slice knots are defined in similar ways, the study of doubly slice knots turns out to be significantly more complicated than the study of slice knots. For this reason, the large number of interesting breakthroughs in the study of slice knots over the last 60 years have not, for the most part, been paralleled in the study of doubly slice knots. For example, one of the most fascinating aspects of 4–dimensional topology is the discrepancy between the smooth and topologically locally flat categories not present in lower dimensions, a discrepancy that has been beautifully captured in the study of slice knots. In the second part of this thesis, we give the first examples demonstrating the presence of this beautiful phenomenon in doubly slice knots. These results also appear in [Mei13].

Theorem C ([Mei13])**.** There exists an infinite family of smoothly slice knots that are topologically doubly slice, but not smoothly doubly slice.

One would like to say that these knots form a subgroup isomorphic to $\mathbb{Z}^\infty$ inside the subgroup of the smooth double concordance group consisting of topologically doubly slice knots, but there is a complication that arises in the definition of doubly slice knots (see Question 3.1.1). On the other hand, we prove the following.

Theorem D ([Mei13])**.** The subgroup of the smooth double concordance group consisting of topologically doubly slice knots contains an infinitely generated subgroup whose members are smoothly slice knots.

These proofs utilize the correction terms coming from Heegaard Floer homology to obstruct certain 3–manifolds from embedding smoothly in S^4. One interesting corollary of this work is the construction of infinitely many rational-homology 3–spheres that embed topologically in S^4, but not smoothly. Heegaard Floer homology is a powerful set of invariants of knots, 3–manifolds, and 4–manifolds introduced by Ozsváth and Szabó (see, for example, [OS03a, OS04a, OS04b]).

Chapter 2

Small Seifert fibered surgeries on hyperbolic Montesinos knots

2.1 Introduction[1]

The study of exceptional surgery on hyperbolic knots has been well developed over the last quarter century. One particularly well studied problem is that of exceptional surgery on arborescent knots, which include Montesinos knots and pretzel knots. Thanks to the positive solution to the Geometrization conjecture [Per03a, Per03b, Per02], any exceptional surgery is either reducible, toroidal, or a small Seifert fibered space. Exceptional surgeries on hyperbolic arborescent knots of length 4 or greater have been classified [Wu11b], as have exceptional surgeries on hyperbolic 2-bridge knots [BW01]. It has been shown that no hyperbolic arborescent knot admits a reducible surgery [Wu96], and toroidal surgeries on hyperbolic arborescent knots of length three are completely classified [Wu11a].

Therefore, it only remains to understand small Seifert fibered surgeries on Montesinos knots of length three. Furthermore, finite surgeries on Mon-

[1]Portions of this chapter have been previous published: Jeffrey Meier, *Small Seifert fibered surgery on hyperbolic pretzel knots*, Algebraic & Geometric Topology **14** (2014), no. 1, 439-487.

tesinos knots only occur in two instances, along three slopes of the pretzel knot $P(-2,3,7)$ and along two slopes of the pretzel knot $P(-2,3,9)$ [FIK$^+$09, IJ09]. Thus, one must only consider non-finite, atoroidal Seifert fibered surgeries on hyperbolic Montesinos knots of length three.

According to Wu [Wu10, Wu11c], the only hyperbolic Montesinos knots of length three that are pretzel knots and might admit Seifert fibered surgeries have the form $P(q_1, q_2, q_3)$ or $P(q_1, q_2, q_3, -1)$, where $(|q_1|, |q_2|, |q_3|) = (2, |q_2|, |q_3|)$, $(3, 3, |q_3|)$, or $(3, 4, 5)$, and in the length four case, then $q_i > 0$ for $i = 1, 2, 3$. Recently, it was shown that hyperbolic pretzel knots of the form $P(p, q, q)$ with p, q positive [IJ11] or $P(-2, p, p)$ with p positive [IJK11] do not admit Seifert fibered surgeries.

Further work by Wu [Wu10, Wu11b, Wu11c] tells us that if a non-pretzel Montesinos knot admits a small Seifert fibered surgery, then it has one of the following forms: $K[1/3, -2/3, 2/5]$, $K[-1/2, 1/3, 2/(2a+1)]$ for $a \in \{3, 4, 5, 6\}$, or $K[-1/2, 1/(2q+1), 2/5]$ for $q \geq 1$.

In this chapter, we address the issue of which of the above listed Montesinos knots admit small Seifert fibered surgeries. The main results are stated below. Keep in mind that there is an orientation reversing homeomorphism $K(\alpha) = \overline{K}(-\alpha)$, where $\overline{K}$ is the mirror of K. Thus, we often consider in our analysis, and present in our results, only one representative of $\{K, \overline{K}\}$.

For the following theorem, recall that the pretzel knot $P(p, q, r)$ with $|p|, |q|, |r| \geq 2$ is hyperbolic unless it is either $P(-2, 3, 3)$ or $P(-2, 3, 5)$, in which case it is the torus knot $T(3,4)$ or $T(3,5)$, respectively ([Oer84]). Below, when we consider the knots $P(-2,2p+1,2q+1)$, we will assume that $|p|<|q|$ when p and q have the same sign and that $p>0$ when their signs differ.

Theorem 2.1.1. The hyperbolic pretzel knot $P(-2,2p+1,2q+1)$, with the conventions discussed above, admits a small Seifert fibered surgery if and only if $p=1$, in which case it admits precisely the following small Seifert fibered surgeries:

- $P(-2,3,2q+1)(4q+6)=S^2(1/2,-1/4,2/(2q-5))$
- $P(-2,3,2q+1)(4q+7)=S^2(2/3,-2/5,1/(q-2))$

Theorem 2.1.2. Hyperbolic pretzel knots of the form $P(3,3,m)$ or $P(3,3,2m,-1)$ admit no small Seifert fibered surgeries. Pretzel knots of the form $P(3,-3,m)$, with $m>1$, admit small Seifert fibered surgeries precisely in the following cases:

- $P(3,-3,2)(1)=S^2(1/3,1/4,-3/5)$
- $P(3,-3,3)(1)=S^2(1/2,-1/5,-2/7)$
- $P(3,-3,4)(1)=S^2(-1/2,1/5,2/7)$
- $P(3,-3,5)(1)=S^2(2/3,-1/4,-2/5)$
- $P(3,-3,6)(1)=S^2(1/2,-2/3,2/13)$

Theorem 2.1.3. The pretzel knots $P(3, \pm 4, \pm 5)$ and $P(3, 4, 5, -1)$ admit no small Seifert fibered surgeries.

Theorem 2.1.4. Suppose that K is a non-pretzel Montesinos knot and $K(\alpha)$ is a small Seifert fibered space. Then either $K = K[-1/2, 2/5, 1/(2q+1)]$ for some $q \geq 5$, or K is on the following list and has the described surgeries.

- $K[1/3, -2/3, 2/5](-5) = S^2(1/4, 2/5, -3/5)$
- $K[-1/2, 1/3, 2/7](-1) = S^2(1/3, 1/4, -4/7)$
- $K[-1/2, 1/3, 2/7](0) = S^2(1/2, 3/10, -4/5)$
- $K[-1/2, 1/3, 2/7](1) = S^2(1/2, 1/3, -16/19)$
- $K[-1/2, 1/3, 2/9](2) = S^2(1/2, -1/3, -3/20)$
- $K[-1/2, 1/3, 2/9](3) = S^2(1/2, -1/5, -3/11)$
- $K[-1/2, 1/3, 2/9](4) = S^2(-1/4, 2/3, -3/8)$
- $K[-1/2, 1/3, 2/11](-2) = S^2(-2/3, 2/5, 2/7)$
- $K[-1/2, 1/3, 2/11](-1) = S^2(-1/2, -2/7, 2/9)$
- $K[-1/2, 1/3, 2/5](3) = S^2(1/2, -1/3, -2/15)$
- $K[-1/2, 1/3, 2/5](4) = S^2(1/2, -1/6, -2/7)$
- $K[-1/2, 1/3, 2/5](5) = S^2(-1/3, -1/5, 3/5)$

- $K[-1/2, 1/5, 2/5](7) = S^2(1/2, -1/5, -2/9)$
- $K[-1/2, 1/5, 2/5](8) = S^2(-1/4, 3/4, -2/5)$
- $K[-1/2, 1/7, 2/5](11) = S^2(-1/3, 3/4, -2/7)$

Each of the theorems stated above is proved below using a common procedure. First, we exploit the symmetries of the Montesinos knots in question to describe the surgery space as a branched double cover of a link. Next, we use rational tangle filling theory and exceptional surgery bounds to restrict our attention to a finite list of such links, i.e, we restrict the parameters for which the Montesinos knots in question can admit small Seifert fibered surgeries. Finally, we use knot theory invariants to show that the branched double covers of links on this finite list cannot be Seifert fibered (excepting, of course, the cases that are). This last step makes use of the Mathematica® package KnotTheory‘ [Wol99].

It should be noted that, concurrent with the preparation of this chapter, the author learned that similar results had been obtained by Wu, though using different techniques. Wu also restricts the families to finite families of surgery spaces, but does so by studying exceptional surgery on *tubed* Montesinos knots (see [Wu12a]). He then appeals to the computer program *Snappex*, to determine the hyperbolic structure of the surgeries in question (see [Wu12c]).

Organization. Section 2.2 presents general background material and outlines how knot invariants will be used to obstruct small Seifert fibered surgeries.

Sections 2.3, 2.4, 2.5, and 2.6 present, respectively, the proofs of Theorems 2.1.1, F, 2.1.3, and 2.1.4.

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2.1.1 A word on non-integral surgeries

In a survey by Wu [Wu98a], it is shown how techniques and results from [Bri98, Wu98b] can be combined with work of Delman [Del95] to study which length three Montesinos knots have exteriors that admit persistent essential laminations.

Theorem 2.1.5. Let K be a hyperbolic Montesinos knot of length three. Then the exterior of K admits a persistent essential lamination, and, thus, cannot admit a non-integral small Seifert fibered surgery, unless $K = K[x, 1/p, 1/q]$ (or its mirror image), where $x \in \{-1/(2n), -1 \pm 1/(2n), -2 + 1/(2n)\}$, and p, q, and n are positive integers.

With this in mind, for many of the families of pretzel knots considered in this paper, it is only necessary to consider integral surgeries. However, for some families, it is necessary to consider non-integral surgeries. To be specific, of all the pretzel knots considered in this paper, only the following families could potentially admit non-integral small Seifert fibered surgeries:

- $P(-2, 2p+1, 2q+1)$ with $1 \leq p < q$
- $P(3, 3, -2m)$ with $m \geq 2$
- $K[-1; 1/3, 1/3, 1/2m]$ with $m \geq 1$
- $P(3, -4, 5)$ or $P(3, 4, 5, -1)$

Thus, whenever such a family is considered, we have shown that, in fact, there are no non-integral small Seifert fibered surgeries. One of the biggest open problems in the study of exceptional Dehn surgery is the following conjecture (see [Gor09b]).

Conjecture 2.1.6. Any Seifert fibered surgery on a hyperbolic knot is integral.

The results of this paper are the final steps of an affirmative answer to Conjecture 2.1.6 in the case of hyperbolic arborescent knots.

Theorem 2.1.7. Any Seifert fibered surgery on a hyperbolic arborescent knot is integral.

2.1.2 A note of the classification for Montesinos knots

After the results in this chapter were announced, Ichihara and Masai [IM13] proved that the knots $K[-1/2, 2/5, 1/(2q+1)]$ admit no small Seifert fibered surgeries for $q \geq 5$. When taken together with the present work, this result completes the classification of exceptional surgeries on arborescent knots.

2.2 Preliminaries

2.2.1 Dehn surgery

Let K be a knot in S^3, and let $N(K)$ be a regular neighborhood of K. Let $M_K = \overline{S^3 \backslash N(K)}$ be the exterior of K. The set of isotopy classes of simple closed curves on $\partial N(K) = \partial M_K$ is in bijection with $H_1(\partial M_K)$, the latter of which is naturally generated by two elements: $[\mu]$ and $[\lambda]$, where $[\mu]$ generates $H_1(M_K) \cong \mathbb{Z}$, $[\lambda] = 0 \in H_1(M_K)$, and μ and λ intersect geometrically once on ∂M_K. Orient μ and λ so that $\mu \cdot \lambda = +1$. The unoriented isotopy class of a simple closed curve $\gamma \subset \partial M_K$ is called a *slope* and can be thought of as an element $m/l \in \mathbb{Q} \cup \{\infty\}$, where $[\gamma] = m[\mu] + l[\lambda]$ in $H_1(\partial M_K)$. The curves μ and λ are called the *meridian* and the *longitude*, respectively.

Given two slopes α and β on T^2, let the *distance* between α and β, $\Delta(\alpha, \beta)$ be their minimal geometric intersection number. If $\alpha = m/l$ and $\beta = m'/l'$, then we have $\Delta(\alpha, \beta) = |ml' - m'l|$.

Let V be a solid torus, and let $\varphi : \partial V \to \partial M_K$ be a homeomorphism

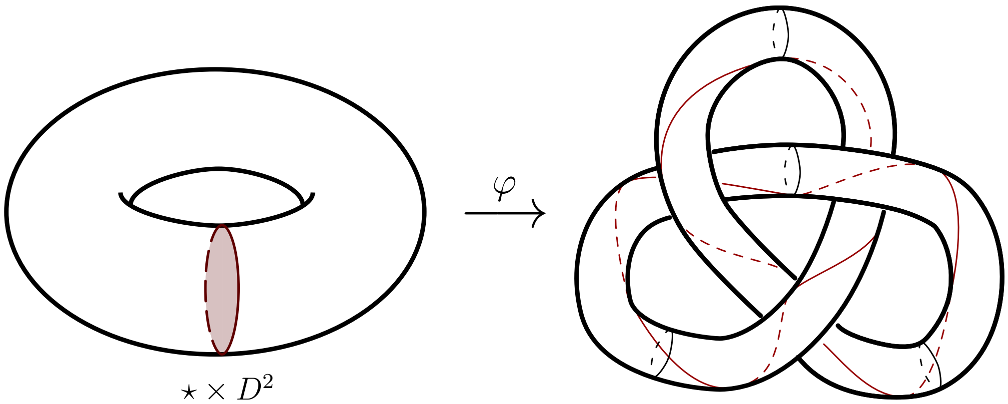


Figure 2.2.1: On the right, we see the exterior, M_K, of the left-handed trefoil. The surgery space $K(0)$ is formed by filling the boundary of M_K with a solid torus such that the meridian maps to a 0-slope (a longitude of K) on ∂M_K.

which takes the meridian of V to a slope α on ∂M_K. Then *Dehn surgery on K along α*, or *α-Dehn surgery on K*, is the space $K(\alpha) = M_K \cup \varphi V$. See Figure 2.2.1. For a general overview of the theory of Dehn surgery, a subject that has been well-studied since its introduction by Dehn in 1910 [Deh10], see [Gor09b].

Dehn surgery generalizes nicely to manifolds M with a torus boundary component $T \subset \partial M$, where M may not be the complement of knot in S^3. Let $\alpha \subset T$ be a slope, then *α-Dehn filling of M on T* is the space $M(\alpha) = M \cup_\varphi V$, where $\varphi : \partial V \to T$ sends the meridian of V onto α. One difference in this scenario is that there may be no canonical way to distinguish a longitude on T, however, $\Delta(\alpha, \beta)$ is still well-defined for any pair of slopes, α and β.

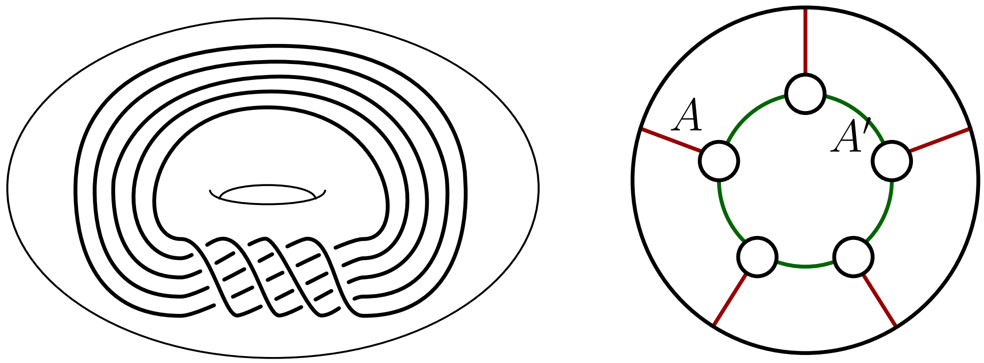


Figure 2.2.2: On the left, we see a $(4, 5)$-curve J inside a solid torus, V, and, on the right, we see a cross section of $V - N(J)$, along with two interesting annuli, A and A'.

2.2.2 Cable spaces

Let V be a solid torus, and let J be a (p, q)-curve inside V (see Figure 2.2.2). The *cable space*, $C(p, q)$, is the space formed by removing a regular neighborhood of J. Let $T_1 = \partial V$ and $T_0 = \partial N(J)$. There is a properly embedded annulus, A, connecting the two boundary components such that $A \cap T_1$ is a p/q-curve (in terms of the standard meridian and longitude on V) and $\gamma = A \cap T_0$ is a $pq/1$-curve (see Figure 2.2.2). Let μ and λ be some choice of meridian and longitude for T_0. Then the slope γ is called the *cabling slope* for $C(p, q)$.

Let A' be a properly embedded annulus such that $A' \cap T_0$ is two $pq/1$-curves, parallel to each other and to γ (see Figure 2.2.2). Now, let $C(p, q)(\alpha)$ denote α-Dehn filling on T_0. Then, if $\alpha = \gamma$, this filling has the effect of capping off A' to form a separating 2-sphere, S, and capping off one boundary component of A to form a disk, D, which intersects T_1 in a p/q-curve. The result is that $C(p,q)(\gamma) = (S^1 \times D^2)\#L(q,p)$.

Let $t : C(p,q) \to C(p,q)$ represent Dehn twisting along A. Then, $t^l(\mu) = \mu + l(pq\mu + \lambda)) = (lpq+1)\mu + l\lambda$. Since $C(p,q)(\mu) = S^1 \times D^2$, it follows that $C(p,q)(t^l(\mu)) = S^1 \times D^2$. So, slopes of the form $(lpq+1)/l$ all correspond to surgery slopes on T_0 that yield solid tori.

This shows that cable spaces have infinitely many fillings returning solid tori, all at distance one from the cabling slope.

On the other hand, we have the following lemma, which follows from the Cyclic Surgery Theorem [CGLS87] and work of Gabai [Gab89]. See [Kan10] for a proof and more general discussion.

Lemma 2.2.1. (a) Let $M \neq T^2 \times I$ be an irreducible and ∂-irreducible 3-manifold with a torus boundary component, T_0. Let T_1 be an incompressible torus in M, distinct from T_0. If α and β are slopes on $T_0 \subset \partial M$ with $\Delta(\alpha,\beta) \geq 2$, such that T_1 is compressible in $M(\alpha)$ and $M(\beta)$, then M is a cable space with cabling slope γ such that $\Delta(\alpha,\gamma) = \Delta(\beta,\gamma) = 1$.

(b) Let $M \neq T^2 \times I$ be an irreducible and ∂-irreducible 3-manifold with a torus boundary component, T_0. Let T_1 be an incompressible torus in M, distinct from T_0. If α and β are slopes on $T_0 \subset \partial M$ with $\Delta(\alpha,\beta) = 1$, such that T_1 is compressible in $M(\alpha)$ and $M(\beta)$, then either

(a) M is a cable space with cabling slope α or β, or

(b) M is the exterior of a braid in a solid torus, $M(\alpha)$ and $M(\beta)$ are solid tori, and $\Delta(\eta_\alpha, \eta_\beta) \geq 4$, where η_α and η_β are the induced slopes of the meridian on T_0.

2.2.3 Seifert fibered spaces

A *fibered solid torus* is formed by gluing the ends of $D^2 \times I$ together with a twist ρ through $\frac{2\pi p}{q}$, where $q \geq 1$ and p and q are relatively prime. There are two types of *fibers*: the *central fiber*, i.e., the image of $(0,0) \times I$ after gluing, and the union of the arcs $x \times I, \rho(x) \times I, \cdots, \rho^{q-1}(x) \times I$, for $x \neq (0,0)$.

A *Seifert fibered space* is a 3-manifold that can be decomposed as a disjoint union of circles (called *fibers*), where each fiber has a regular neighborhood homeomorphic to a fibered solid torus, i.e., the fiber becomes the central fiber of the fibered solid torus. Viewing the neighborhood this way, if $q = 1$, we say the fiber is *ordinary*. If $q \geq 2$, we say the fiber is *exceptional with multiplicity* q. In the latter case, the fibers surrounding the central fiber are called (p, q)-*curves*.

If M is a Seifert fibered space, there is a natural projection $\pi : M \to \Sigma$ that identifies each fiber to a point. The surface Σ is called the base space. We can record the exceptional fiber information in the form of cone points on Σ, so M is a circle bundle over the resulting orbifold. Another way to recover M is to remove a disk neighborhood of each cone point on Σ and cross the resulting surface with S^1. The result is a manifold with torus boundary components. If we choose meridian and longitude coordinates for each boundary component so that the projection of the meridians to the base surfaces is one-to-one onto the boundary of the removed disks and the longitude is $\star \times S^1$ in the circle product, then M is the result of Dehn filling on the boundary components along the slopes p'/q, where $pp' \equiv 1 \pmod q$. If M is a Seifert fibered space with base space Σ and n exceptional fibers with fibered solid torus neighborhoods consisting of (p_i/q_i)-curves for $i = 1, 2, \ldots, n$, we write $M = \Sigma(p'_1/q_1, \ldots, p'_n/q_n)$, or sometimes $M = \Sigma(q_1, \ldots, q_n)$. In fact, the homeomorphism type of M is determined by Σ and the *Seifert invariants*: $\{p'_1/q_1, \ldots, p'_n/q_n\}$, up to permutation, and up to the relation $\{p'_1/q_1, p'_2/q_2, \ldots, p'_n/q_n\} = \{p'_1/q_1 \pm 1, p'_2/q_2 \mp 1, \ldots, p'_n/q_n\}$. In other words, $\sum_{i=1}^{n} p'_i/q_i$ is an invariant of M. Because of this, it is often useful to standardize the notation so that the Seifert invariants are all positive and less than one. To do this, we subtract out the integer part of each fraction and collect them in a single term, b. We write $M = \Sigma(b; p'_1/q_1, \ldots, p'_n/q_n)$, where $0 < p'_i < q_i$ and $b \in \mathbb{Z}$.

A Seifert fibered space is called *small* if the base space is a sphere and the number of exceptional fibers is at most three.

Next, we recall a fact about Dehn filling on Seifert fibered manifolds that will be useful throughout this paper. Let M be a Seifert fibered manifold with a torus boundary component $T \subset \partial M$. The fibering of M induces a fibering of T, and the slope, γ, of the induced fibers on T is called the *Seifert slope* of T. Now, the Seifert fibering of M will extend to a Seifert fibering of α-Dehn filling on M provided that $\alpha \neq \gamma$. In fact, we have the following. See [Hei74] for a complete treatment of Dehn filling on Seifert fibered spaces with boundary.

Lemma 2.2.2. If M is a Seifert fibered manifold with base surface Σ and n exceptional fibers, $T \subset \partial M$ is a torus boundary component (corresponding to a circle boundary component $C \subset \partial\Sigma$), and γ is the Seifert slope T, then let $M(\alpha)$ denote α-Dehn filling on T, let $d = \Delta(\alpha, \gamma)$, and let $\hat{\Sigma} = \Sigma \cup_C D^2$. Then,

(a) If $d \geq 2$, $M(\alpha)$ is a Seifert fibered space with base surface $\hat{\Sigma}$ and $n+1$ exceptional fibers (the original exceptional fibers, plus a new one of multiplicity d).

(b) If $d = 1$, $M(\alpha)$ is a Seifert fibered space with base surface $\hat{\Sigma}$ and (the original) n exceptional fibers.

(c) If $d = 0$, $M(\alpha) = N \# L$, where N is a Seifert fibered space with base surface $\hat{\Sigma}$ and (the original) n exceptional fibers, and L is a Lens space.

As an example, consider $D^2(a, b)$ with Seifert slope r/s, and let $d = \Delta(m/l, r/s) = |ms - lr|$. Then (as developed in [Gor09b]),

$$D^2(a,b)(m/l) = \begin{cases} S^2(a,b,d) & \text{if } d \geq 2 \\ L(m, lb^2) & \text{if } d = 1 \\ L(a,b) \# L(b,a) & \text{if } d = 0 \end{cases}$$

2.2.4 Montesinos knots

A *tangle* is a pair (B, A), where $B \cong B^3$ and A is a pair of properly embedded arcs in B. A *marked tangle* is a tangle along with an identification of its boundary $\partial(B, A) = (S, S \cap A)$, which is a 2-sphere with 4 distinguished points, with the pair $(S^2, \{NE, NW, SW, SE\})$. The *trivial tangle* is the tangle which is homeomorphic as a marked tangle to $(D^2, \{2\ points\}) \times I$. Let h and r be the tangle operations where h adds a positive horizontal half-twist (right-handed), and r is reflection in the (NW/SE)-plane.

Let $[c_1, c_2, \ldots, c_m]$ be a sequence of integers, and let $p/q = \frac{1}{c_1 + \frac{1}{c_2 + \frac{1}{\cdots + c_m}}}$. The *rational tangle*, $\mathcal{R}(p/q)$ is formed by applying the operation $(h^{c_m} r)(h^{c_{m-1}} r) \cdots (h^{c_1} r)$ to the trivial tangle, which we denote $\mathcal{R}(1/0)$. Note that, as an unmarked tangle, $\mathcal{R}(p/q)$ is trivial, one can just untwist it. On the other hand, Conway showed [Con70] that, as marked tangles, $\mathcal{R}(p/q) = \mathcal{R}(p'/q')$ if and only if $p/q = p'/q'$.

A *Montesinos link of length* n is a link formed by connecting n rational tangles to each other in a standard fashion. We denote such a knot by $K[p_1/q_1, \ldots, p_n/q_n]$ (see Figure 2.2.3). In the special case where each $p_i = \pm 1$, we have what is called a *pretzel knot*. In this case, each tangle is just a strand of vertical twists, since $1/q$ has the continued fraction expansion $[q]$. It is easy to see that Montesinos links of length one or two are the same. These links are called 2-*bridge links*, and will be denoted $K[p/q]$, where p/q is the rational number describing the tangle twists.

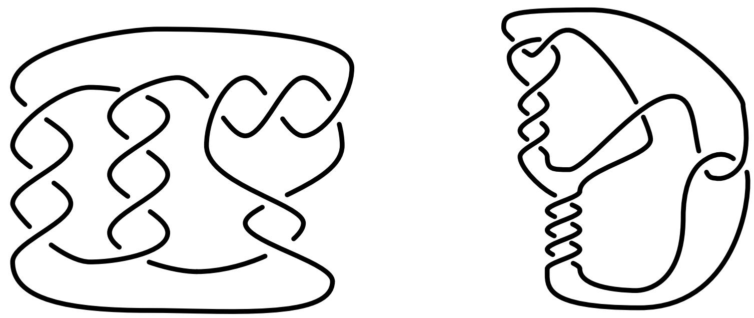


Figure 2.2.3: Above we have the Montesinos knot $K[1/3, 1/4, -3/5]$ and the 2-bridge knot $K[43/95]$ (with continued fraction $[2, 4, 1, 3, 2]$).

Montesinos links of length three are determined up to the same relations as Seifert fibered spaces, but when $n > 3$, the cyclic order of the strands also matters. In either case, we can normalize the invariants and write $K[b; p_1/q_1, \ldots, p_n/q_n]$ where $0 < p_i < q_i$ and $b \in \mathbb{Z}$. In fact, we have the following proposition, which follows from Theorem 2.2.5 below.

Proposition 2.2.3. The double cover of S^3, branched along the Montesinos link $K[p_1/q_1, \ldots, p_n/q_n]$, is the Seifert fibered space $S^2(p_1/q_1, \ldots, p_n/q_n)$.

We remark that it is often helpful to allow p_i/q_i to be zero, ∞, or 1 for some i, in either the notation for Montesinos links or Seifert fibered spaces. For our purposes, this will only happen when the length n is three or less, and the result should be clear from the context. For example, $K[1/3, -1/2, 1/0]$ is the connected sum of a trefoil knot and a Hopf link, $K[1/3, 2/7, 0] = K[2/13]$, and $S^2(2, 3, 1)$ is a lens space.

2.2.5 Seifert fibered surgery on knots with symmetries

In this section, we recall some known results about Seifert fibered surgery on knots that admit a strong inversion, have period two, or both. In what follows, let $K \subset S^3$ be a knot and let $\varphi : S^3 \to S^3$ be a non-trivial orientation preserving involution such that $\varphi(K) = K$ and $C_\varphi = \mathrm{Fix}(\varphi) \neq \emptyset$. By the positive solution to the Smith conjecture, C_φ is an unknotted circle in S^3 [MB84].

Definition 2.2.4. If $C_\varphi \cap K \neq \emptyset$, then φ is called a *strong inversion* of K and K is called *strongly invertible.* In this case, $C_\varphi \cap K = 2$ points and φ reverses the orientation of K.

If $C_\varphi \cap K = \emptyset$, then we say φ is a *cyclic symmetry of order 2* and that K has *period* 2.

In this paper, we will only be interested in strong inversions and cycles of period 2. For a more general treatment of Dehn surgery on knots with symmetries, see [Mot03].

First, let us consider strongly invertible knots. Let $K \subset S^3$ be a knot with a strong inversion φ. Then φ restricts to an involution of the knot exterior, M_K, and the quotient of M_K by the action of φ is a tangle, $\mathcal{T}_K$. The well-known Montesinos trick gives a correspondence between Dehn filling on M_K and rational tangle filling on $\mathcal{T}_K$. For details, see [Gor09b]. The following is originally due to Montesinos [Mon75].

Theorem 2.2.5. Let $\mathcal{T}$ be a marked tangle. Then $\widetilde{\mathcal{T}}(r/s) \cong \widetilde{\mathcal{T}(-r/s)}$.

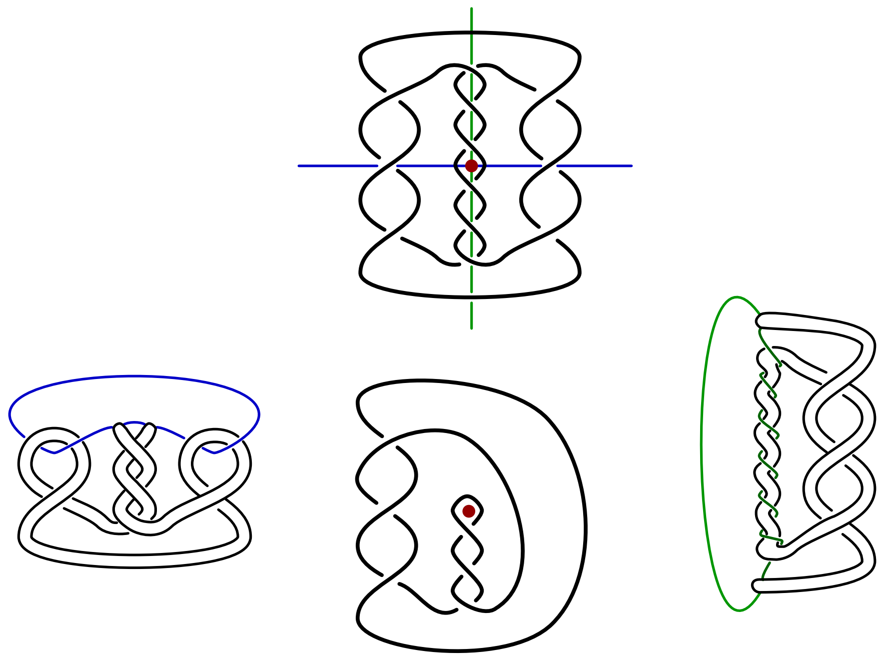


Figure 2.2.4: The knot $P(3, 3, -6)$, shown with its three symmetries and the resulting quotients.

Let $L_{r/s} = \mathcal{T}_K(-r/s)$, so $L_{r/s}$ is a knot or a two-component link in S^3 with $K(r/s)$ as the double cover of S^3, branched along $L_{r/s}$. Suppose that $K(r/s)$ is a small Seifert fibered space. Let $\bar{\varphi} : K(r/s) \to K(r/s)$ be the involution induced by extending φ across the filling solid torus. Then $K(r/s)/\bar{\varphi} = S^3$.

If K is not a trefoil knot, then we can assume that $\bar{\varphi}$ is fiber-preserving [Mot03]. Let $\pi : K(r/s) \to S^2$ be the Seifert fibration of $K(r/s)$. Let $C_{\bar{\varphi}} = \mathrm{Fix}(\bar{\varphi})$. If each component of $C_{\bar{\varphi}}$ is a fiber in $K(r/s)$, then $K(r/s)\backslash C_{\bar{\varphi}}$ admits a Seifert fibered structure. Since this structure is compatible with $\bar{\varphi}$, $S^3\backslash L_{r/s}$ admits a Seifert fibered structure. In other words, $L_{r/s}$ is a *Seifert link.*

Let $\hat{\varphi} : S^2 \to S^2$ be the induced involution of the base orbifold. If one component of $C_{\bar{\varphi}}$ is not a fiber in $K(r/s)$, then $\hat{\varphi}$ is reflection across the equatorial circle, $C_{\hat{\varphi}}$, of S^2 and all of the cone points lie on $C_{\hat{\varphi}}$ [Mot03]. In this case, $L_{r/s} = C_{\bar{\varphi}}/\bar{\varphi}$ is a length three Montesinos link [MM02]. So, we have the following, as stated in [IJ11].

Proposition 2.2.6. Let K be a strongly invertible hyperbolic knot, and let $r/s \in \mathbb{Q}$. Let $L_{r/s}$ be the link obtained by applying the Montesinos trick to $K(r/s)$. If $K(r/s)$ is a small Seifert fibered space with base orbifold S^2, then $L_{r/s}$ is either a Seifert link or a Montesinos link.

Seifert links are well understood [BM70, EN85]. In the present paper, we will only be concerned with Seifert knots and Seifert links with two components, in which case we have the following.

Lemma 2.2.7. Let $L \subset S^3$ be a Seifert link with at most two components. Then L is equivalent to one of the following:

(a) A torus knot

(b) A two-component torus link

(c) A two-component link consisting of a torus knot together with a core curve of the torus on which it lies.

Note that, in particular, every component of a Seifert link is a torus knot or an unknot.

Now let $K \subset S^3$ be a knot with a cycle symmetry φ of order 2. Suppose that $K(r/s)$ is a Seifert fibered space with base surface S^2, and let $\bar{\varphi}$ be the extension of $\varphi|_{M_K}$ to $K(r/s)$. Then, $K(r/s)$ has a $\bar{\varphi}$-invariant Seifert fibered structure [MM02]. Let $C_{\bar{\varphi}} = \text{Fix}(\bar{\varphi})$, and let $L_{r/s} = C_{\bar{\varphi}}/\bar{\varphi}$.

If r is odd, then $L_{r/s}$ is a knot. If r is even, then $L_{r/s}$ is a link. Let $K_\varphi = K/\varphi$. K_φ is called the *factor knot of* K (with respect to φ), and let $C_\varphi = \text{Fix}(\varphi)$. In the case where r is odd, we can view $L_{r/s}$ as the image of C_φ/φ after $r/2s$ surgery on K_φ, so $L_{r/s}$ is a knot in $K_\varphi(r/2s)$. If r is even, then $L_{r/s}$ is the image of C_φ/φ in $K_\varphi(r/2s)$ together with the core of the surgery torus, so $L_{r/s}$ is a link in $K_\varphi(r/2s)$.

Let $\pi : K(r/s) \to S^2$ be a Seifert fibering of $K(r/s)$, and let $\hat{\varphi}$ be the induced involution of S^2, with fixed point set $C_{\hat{\varphi}}$. In [MM02], it is shown that if K is not a torus knot or a cable of a torus knot, then no component of $C_{\bar{\varphi}}$ is

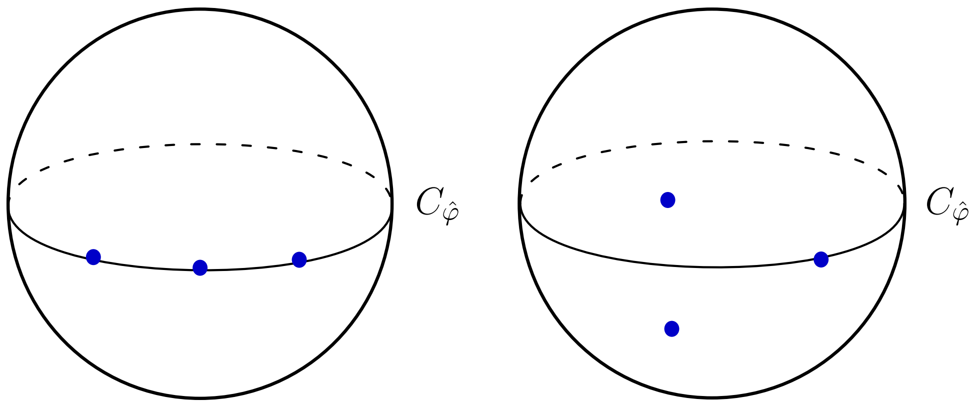


Figure 2.2.5: Possible configurations of cone points in the base sphere of a small Seifert fibered space

a fiber in $K(r/s)$ and $C_{\hat{\varphi}}$ is the equatorial circle in S^2. This implies that $\hat{\varphi}$ is reflection across the equator. Since $\bar{\varphi}$ is fiber preserving, $\hat{\varphi}$ must pair up cone points in the northern hemisphere with cone points in the southern hemisphere. Let k denote the number of cone points in the northern hemisphere. For our purposes, $k = 0$ or $k = 1$. From [MM02], we have the following:

Lemma 2.2.8. 1. If $k = 0$, then $K_\varphi(r/2s) = K(r/s)/\bar{\varphi} \cong S^3$.

2. If $k = 1$, then $K_\varphi(r/2s) = K(r/s)/\bar{\varphi}$ is a lens space..

Note that S^3 and $S^2 \times S^1$ are not lens spaces. These facts can be helpful in obstructing Seifert fibered surgeries, based on the knot type of K_φ. Throughout, U will represent the unknot.

Corollary 2.2.9. Let $K \subset S^3$ be a period 2 hyperbolic knot with factor knot K_φ. Suppose $K(r/s)$ is a small Seifert fibered space.

1. If $K_\varphi = T_{p,q}$ and r is even, then $\Delta(pq, r/2s) = 1$, so $|r - 2spq| = 2$.

2. If $K_\varphi = T_{p,q}$ and r is odd, then $\Delta(pq, r/2s) = 1$, so $|r - 2spq| = 1$.

3. If $K_\varphi = U$ and $k = 0$, then $|r| \leq 2$.

4. If $K_\varphi = U$ and $k = 1$, then $|r| \geq 3$.

Proof. If $K(r/s)$ is a small Seifert fibered space, then $K_\varphi(r/2s)$ is a lens space if $k = 1$ and S^3 if $k = 0$. Such surgeries on U and $T_{p,q}$ are well understood (see [Gor09b]). □

2.2.6 Some exceptional Dehn surgery results

There are many important results in the study of exceptional Dehn surgery that give limitations on which slopes can be exceptional for a hyperbolic knot K. Below, we present some of the results that will be used in this paper. First, we state an important result of Lackenby and Meyerhoff [LM08] that tells us that exceptional fillings are always "close" to each other.

Theorem 2.2.10. Suppose M is a hyperbolic manifold with torus boundary component $T \subset \partial M$ and that α and β are exceptional filling slopes on T. Then $\Delta(\alpha, \beta) \leq 8$.

The distance bound of 8 above can be improved if one specifies the type of space for each of $M(\alpha)$ and $M(\beta)$. Let S and T represent the sets of reducible and toroidal manifolds, respectively. Let L represent the set of lens spaces. Let $\Delta(C_1, C_2)$ represent the largest possible value of $\Delta(\alpha, \beta)$ such that there exists a hyperbolic manifold M with $M(\alpha)$ a manifold of type C_1

and $M(\beta)$ a manifold of type C_2 (we will always consider manifolds with one boundary component, though the theory is more general). The following table presents the known values of $\Delta(C_1, C_2)$.

	S	T	S^3	L
S	1	3	?	1
T		8	2	?
S^3			0	1
L				1

Notice that in the case of (S, S^3), this is equivalent to the cabling conjecture, and in the case of (T, L), the bound is known to be either 3 or 4 [Lee11]. For a more thorough discussion of these bounds, the manifolds achieving them, and precise references, see [Gor09b] and [Gor99].

Suspiciously absent from the above table are bounds on the distance between a (non-lens space) small Seifert fibered surgery and the other types of exceptional surgeries. These seem to be the most difficult cases to analyze, and, in particular, it is not known whether or not the distance 8 bound of Lackenby and Meyerhoff can be improved in most of the cases (though, see Theorems 2.2.11 and 2.2.12 below).

The following is a consequence of Corollary 7.6, Proposition 14.1, and Proposition 16.1 in [BGZ12].

Theorem 2.2.11. For any hyperbolic manifold M, if $M(\alpha) = A \cup_{T^2} B$ is toroidal with one of A or B non-Seifert fibered, then for any slope β such that $M(\beta)$ is a Seifert fibered space, $\Delta(\alpha, \beta) \leq 6$.

Since many of the pretzel knots studied below are genus one, it will be helpful for us to have the following result, which gives particularly strong bounds on small Seifert fibered surgery slopes [BGZ11] of such knots.

Theorem 2.2.12. Let K be a hyperbolic knot of genus one such that $K(0)$ is a non-Seifert fibered toroidal manifold. If $K(\alpha)$ is a small Seifert fibered space for some $\alpha \in \mathbb{Q}$, then $\Delta(\alpha, 0) \leq 3$.

2.2.7 Montesinos links, torus links, and invariants from knot theory

In this section we give a very brief overview of some knot and link invariants and how they will be used to obstruct the quotient links encountered in this paper from being Seifert links or Montesinos links. We will present a series of criteria that will applied in each of the following sections.

A link is called *k-almost alternating* if it has a k-almost alternating diagram, but no $(k-1)$-almost alternating diagram, i.e., if it has a diagram D such that changing k crossings of D gives a new diagram that is alternating, but no such diagram where the same result is achieved after $k-1$ crossing changes.

Recall that the Khovanov homology, $\mathrm{Kh}(L)$, is a bi-graded abelian group associated to L, and that the *width* of $\mathrm{Kh}(L)$ is the number of diagonals that support a nontrivial element in $\mathrm{Kh}(L)$. Denote this width by $|\mathrm{Kh}(L)|$. Then we have the following theorem. (See, for example, [AP04].)

Theorem 2.2.13. Let L be a non-split k-almost alternating link. Then $|\mathrm{Kh}(L)| \leq k + 2$.

It has been shown by Abe and Kishimoto [AK10] that any Montesinos link is either alternating or 1-almost-alternating, so we have our first obstruction criterion.

Criterion 2.2.14. If $|\mathrm{Kh}(L)| \geq 4$, then L is not a length three Montesinos link.

When we encounter links L that do not satisfy this criterion, then we will use the following program to show they are not a Montesinos link. We will generate a list of all Montesinos links whose crossing numbers are compatible with that of L (i.e., less than the number of crossings in a diagram of L). (Note that the crossing number of a Montesinos link is well understood [LT88].) We will then check this list for elements that, if L is a knot, have the same determinant, Alexander polynomial, Jones polynomial, Khovanov homology, and, if need be, Kauffman polynomial or HOMFLYPT polynomial, and that, if L is a 2-component link, have the same determinant, Jones polynomial, Khovanov homology, and if need be, Kauffman polynomial or HOMFLYPT polynomial. We will refer to this method as *Method 1.* This very large number of computations was performed using the KnotTheory‘ package for Mathematica® [Wol99].

Examples of the Mathematica files used to implement Method 1 and to calculate knot invariants throughout this paper are available on the author's webpage, and further information will be provided upon request.

Next, we observe that if K is a length three Montesinos link, then it is the union of 2-bridge knots and unknots. If K is the union of two unknots, then it has the form $K[p_1/q_1, p_2/q_2, p_3/q_3]$, where each p_i is even. If one component of K is the 2-bridge knot $K[p/q]$, then K has the form $K[p_1/q_1, p_2/q_2, x/q]$ with q_1 and q_2 even and with $x = p$ or $\bar{p}$, where $p\bar{p} \equiv 1 \pmod{q}$. If we consider the unknot a 2-bridge knot, then we have the following criterion.

Criterion 2.2.15. If K is a 2-component link such that one component is not a 2-bridge knot, then K is not a Montesinos link.

Using Method 1 and Criteria 2.2.14 and 2.2.15, any knot or link we encounter that we claim is not a Montesinos knot or link is shown to not be a Montesinos knot or link.

Now we recall some facts about torus knots (see, for example, [Cro04]). Let $T(p,q)$ be the (p,q)-torus link for $p > q \geq 2$, where $T(p,q)$ is a knot if and only if p and q are coprime. Then, $T(p,q)$ is a positive link, i.e., has a diagram with all positive crossings. Furthermore, in the case of a torus knot, $2g(T(p,q)) = (p-1)(q-1)$, where $g(K)$ denotes the genus of the knot K, and $\det(T(p,q)) = p$ if q is even, and 1 if both p and q are odd, where $\det(L)$ denotes the determinant of the link L. Let $s(K)$ denote the Rasmussen invariant of K, as defined in [Ras10], where the following was shown.

Proposition 2.2.16. If K is a positive knot, then $s(K) = 2g(K)$.

Recall that $2g(K)$ is bounded below by the breadth of the Alexander polynomial, which we denote $\mathrm{br}(\Delta_K(t))$. This gives us the following criterion.

Criterion 2.2.17. If $s(K) < \mathrm{br}(\Delta_K(t))$ or $2g(K) \neq s(K)$, then K is not a torus knot.

We also have, by our discussion above:

Criterion 2.2.18. If $\det(K) > s(K) + 1$, then K is not a torus knot.

If we consider the unknot a torus knot, then each component of a two component Seifert link is a torus knot, so we have the following criterion.

Criterion 2.2.19. If L is a two-component link such that a component is not a torus knot, then K is not a Seifert link.

In what follows, Criteria 2.2.17, 2.2.18, and 2.2.19 often suffice to prove that a link is not a Seifert link. In the few cases where they fail, further argument is given to accomplish the feat.

2.3 The case of $(2, |q_2|, |q_3|)$

Let $K_{p,q}$ be the hyperbolic pretzel knot $P(-2, 2p+1, 2q+1)$ (see Figure 2.3.1). Since Ichihara and Jong have shown that $K_{p,p}$ admits no small Seifert fibered surgery [IJK11], and by interchanging p and q if necessary, we may assume that $|q| > |p|$ if p and q have the same sign and $p > 0$ otherwise. Let $\alpha_r = 4(p + q + 1) - r$, for $r \in \mathbb{Q}$.

Note that α_r is chosen this way so that α_r-surgery on $K_{p,q}$ will correspond with r-filling of $\mathcal{T}_{p,q}$. This becomes clear if one carefully follows through

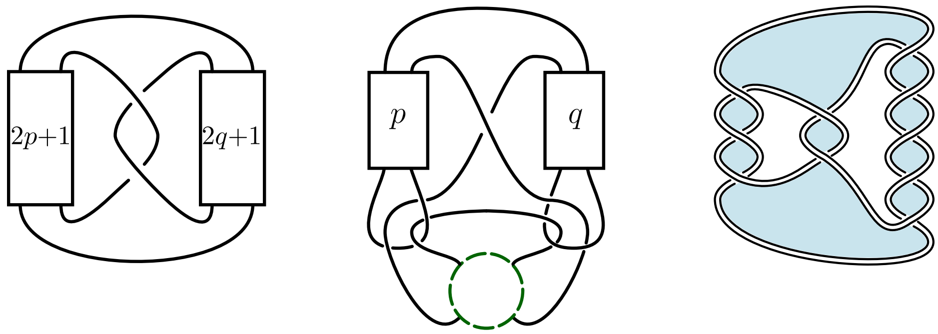


Figure 2.3.1: The pretzel knot $P(-2, 2p+1, 2q+1)$, the quotient tangle $\mathcal{T}_{p,q}$, and the pretzel knot $P(-2, 5, -3)$, shown as the boundary of a punctured Klein bottle

the process of obtaining $\mathcal{T}_{p,q}$ from $K_{p,q}$ by applying the Montesinos trick and isotoping.

Our first result is the following.

Theorem 2.1.1. The hyperbolic pretzel knot $P(-2, 2p+1, 2q+1)$, with the conventions discussed above, admits a small Seifert fibered surgery if and only if $p = 1$, in which case it admits precisely the following small Seifert fibered surgeries:

- $P(-2, 3, 2q+1)(4q+6) = S^2(1/2, -1/4, 2/(2q-5))$
- $P(-2, 3, 2q+1)(4q+7) = S^2(2/3, -2/5, 1/(q-2))$

We remark that the existence of these exceptional surgeries was previously known [EM97].

The key fact in our method of analyzing these knots is that they are strongly invertible. Let $\mathcal{T}_{p,q}$ be the tangle obtained by performing the Mon-

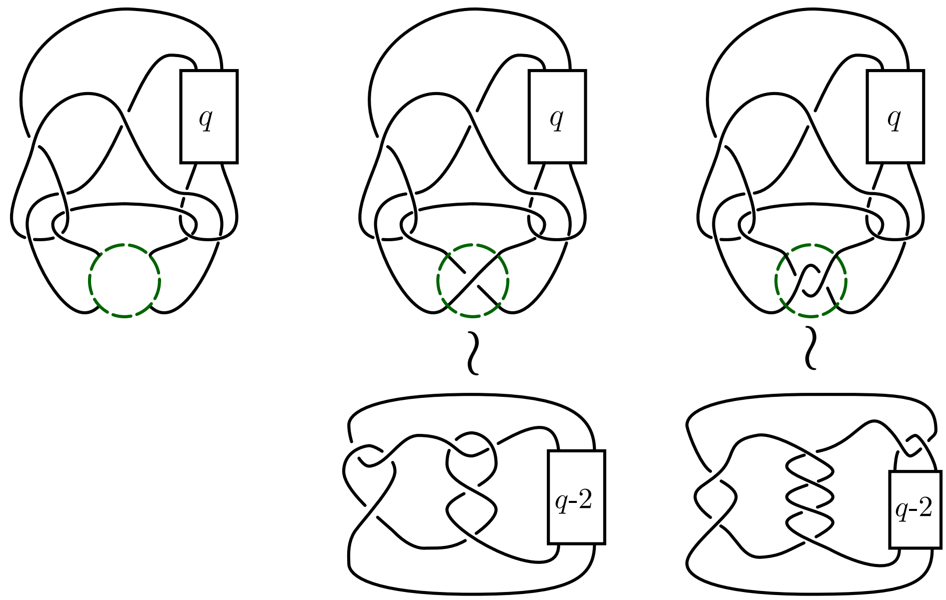


Figure 2.3.2: The tangle $\mathcal{T}_{1,q}$, along with fillings $\mathcal{T}_{1,q}(1)$ and $\mathcal{T}_{1,q}(2)$ and the respective Montesinos links that result after isotopy: $K[2/3, -2/5, 1/(q-2)]$ and $K[1/2, -1/4, 2/(2q-5)]$

tesinos trick (see Figure 2.3.1). We now have the advantage of viewing the surgery space $K_{p,q}(\alpha_r)$ as the branched double cover of S^3 along $\mathcal{T}_{p,q}(r)$. It is easy to verify the two classes of exceptional surgeries in Theorem 2.1.1 by noticing that $\mathcal{T}_{1,q}(1)$ and $\mathcal{T}_{1,q}(2)$ are the Montesinos links $K[2/3, -2/5, 1/(q-2)]$ and $K[1/2, -1/4, 2/(2q-5)]$, respectively. [See Figure 2.3.2.]

The proof that the $K_{p,q}$ admits no other small Seifert fibered surgeries is accomplished by the following two lemmas and the techniques of Subsection 2.2.7.

Lemma 2.3.1. If $K_{p,q}(\alpha_r)$ is a small Seifert fibered space for $p \neq 1$, then $|p| \leq 8$ and $|q| \leq 8$.

Lemma 2.3.2. If $K_{1,q}(\alpha_r)$ is a small Seifert fibered space for $r \notin \{1, 2\}$, then $|q| \leq 8$.

Before we prove these lemmas, we should remark on the possible surgery slopes α_r. We notice that each knot $K_{p,q}$ bounds a punctured Klein bottle at slope α_0 (see Figure 2.3.1). It follows that $K_{p,q}(\alpha_0)$ is toroidal. By Theorem 2.2.10, it follows that if $K_{p,q}(\alpha_r)$ is a small Seifert fibered space, then $\Delta(\alpha_r, \alpha_0) \leq 8$.

In many cases, it should be possible to reduce this distance bound to 5, but this is dependent on work in progress by Boyer, Gordon, and Zhang [BGZ12]. However, using Theorem 2.2.11, we can fairly easily show the following lemma.

Lemma 2.3.3. If $K_{p,q}(\alpha_r)$ is a small Seifert fibered space, and $|p|, |q| \geq 4$, then $\Delta(\alpha_r, \alpha_0) \leq 6$.

Of course, if r is integral, this means that $|r| \leq 6$, and if we have $r/s \in \mathbb{Q}$, we have that $|4(p + q + 1)s - r| \leq 6$.

Proof of Lemma 2.3.3. We begin by noticing that $K_{p,q}(\alpha_0) = D^2(2, 2) \cup_{T^2} X_{p,q}$ (see Figure 2.3.3). Under the hypotheses of the lemma, we will show that $X_{p,q}$ is not a Seifert fibered space, so Theorem 2.2.11 gives us the desired bound. If $p = 1$, then $X_{1,q} = D^2(3, |q - 1|)$, and Theorem 2.2.11 does not apply.

Consider the following fillings on $X_{p,q}$. (See Figure 2.3.4.)

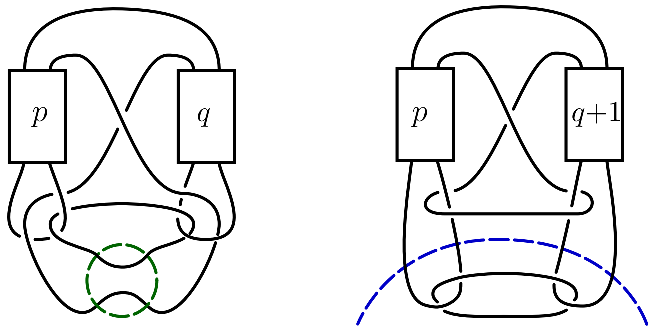


Figure 2.3.3: The link $\mathcal{T}_{p,q}(0)$, whose branched double cover corresponds to α_0-surgery on $K_{p,q}$

$$\begin{aligned}
&X_{p,q}(0) = L(p+q+1,1)\\
&X_{p,q}(\infty) = L(4pq-2p-2q-3, 2pq-q-2)\\
&X_{p,q}(-1) = S^2(1/3, 1/(p-1), q/(q-1))
\end{aligned}$$

If $X_{p,q}$ is a Seifert fibered space, then it has, for its base surface, either D^2 or M^2 (the Möbius band). We will make use of Lemma 2.2.2. If $X_{p,q}$ is Seifert fibered over the disk with more than two exceptional fibers it cannot have lens space fillings. If $X_{p,q}$ has the form of $D^2(a)$, then it cannot have fillings with three exceptional fibers, so $X_{p,q}(-1)$ must be a lens space. This implies that $p = 2$ or $q = 2$. If $X_{p,q}$ has base surface M^2, then it can only have lens space fillings or fillings with at least three exceptional fibers, two of which have multiplicity two. Thus, we must have $p, q = 2, 3$. So, assume $X_{p,q} = D^2(a, b)$.

In this case, $X_{p,q}$ has one reducible filling at slope γ and the property that any lens space filling must be at distance one from γ. By considering the three fillings given above, it follows that $\gamma = 0, \infty$, or ± 1. If $\gamma = -1$, then $X_{p,q}(-1)$ must be reducible, so $p = 1$ or $q = 1$, both of which are not allowed

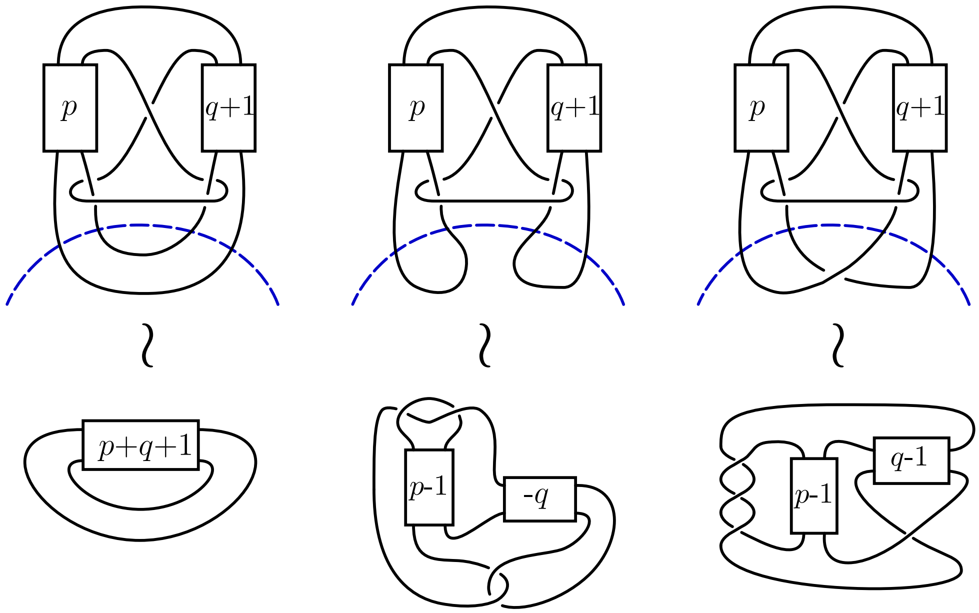


Figure 2.3.4: The three fillings, 0, ∞, and 1, on $\mathcal{T}_{p,q}$ used to show that $X_{p,q}$ is not Seifert fibered.

values. If $\gamma = 0$ or if $\gamma = \infty$, then $X_{p,q}(-1)$ must be a lens space, so $p = 2$ or $q = 2$. Finally, if $\gamma = 1$, then the filling $X_{p,q}(-1)$ is at distance two from the reducible filling, so it must have an exceptional fiber of multiplicity 2. It follows that $p = 3$ or $q = 3$. $\square$

We remark that the lemma could be strengthened to say that $X_{p,q}$ is non-Seifert fibered if and only if $p \neq 1$ by showing that $X_{p,q}(1)$ is neither reducible, a lens space, or a small Seifert fibered space with finite fundamental group, as would need to be the case given the different Seifert fibered structures $X_{p,q}$ might have. However, we will not need anything stronger than what we have proved.

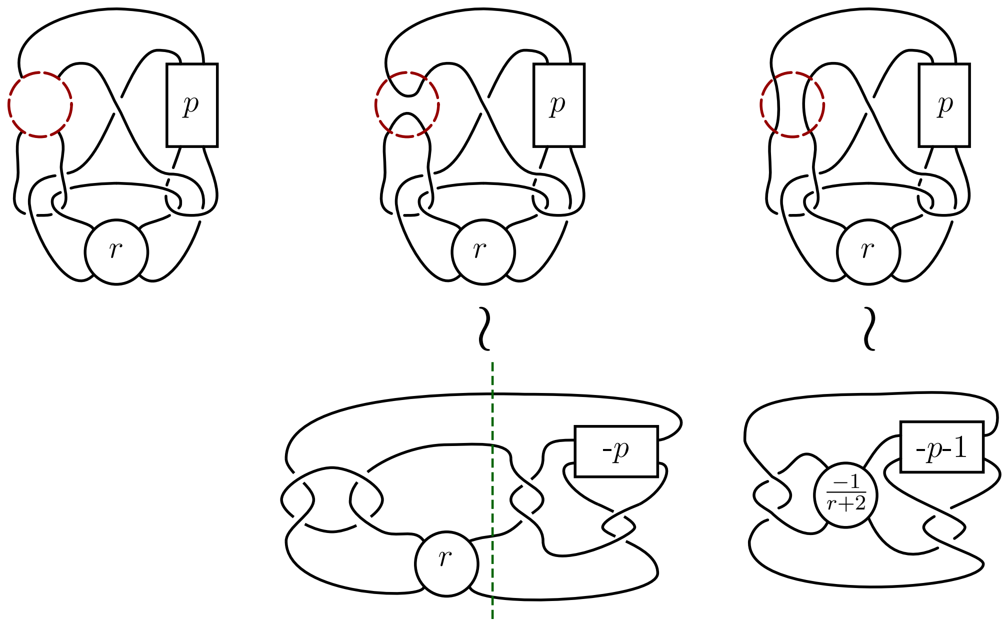


Figure 2.3.5: The tangle $\mathcal{S}_{p,r}$, along with two fillings, $\mathcal{S}_{p,r}(0)$ and $\mathcal{S}_{p,r}(\infty)$, and their equivalents after isotopy.

Proof of Lemma 2.3.1. Suppose that $p \neq 1$ and remove a ball around the q-twists of $\mathcal{T}_{p,q}(r)$ to form the tangle $\mathcal{S}_{p,r}$ (see Figure 2.3.5). Let $N_{p,r}$ denote the branched double cover of $\mathcal{S}_{p,r}$. First, we will show that $N_{p,r}$ is hyperbolic.

We begin by showing some interesting fillings of $N_{p,r}$ (see Figure 2.3.5).

$$N_{p,r}(-1/q) = K_{p,q}(\alpha_r)$$
$$N_{p,r}(0) = D^2(1/2, -1/2) \cup_{T^2} D^2(-1/2, p/(2p-1))$$
$$N_{p,r}(\infty) = T(2, 2p+3)(\alpha_r) = S^2(-1/2, -1/(r+2), -2/(2p+3))$$
$$N_{p,r}(-1) = K[-2p/(6p+1)](\alpha_r)$$

We remark that $N_{p,r}(\infty)$ and $N_{p,r}(-1)$ correspond to (α_r)-surgery on $K_{p,0}$ and $K_{p,-1}$, respectively. The latter is a 2-bridge knot with no exceptional fillings (if $p \neq 1$), according to the classification by Brittenham and Wu

([BW01]). Thus, $N_{p,r}(-1)$ is hyperbolic if $p \neq 1$.

Now, suppose that $N_{p,r}$ is not hyperbolic, so it must be reducible, ∂-reducible, Seifert fibered, or toroidal by geometrization. However, $N_{p,r}$ cannot be reducible, since it has two distinct irreducible fillings at slopes 0 and -1 (this follows from the solution to the knot complement problem [GL89]). Similarly, it cannot be Seifert fibered, since it has a hyperbolic filling at slope -1. It follows that $N_{p,r}$ cannot be ∂-reducible, since the only irreducible, ∂-reducible manifold with torus boundary is Seifert fibered, namely, the solid torus.

Finally, suppose that $N_{p,r}$ is toroidal. If any essential torus were non-separating, then all fillings of $N_{p,r}$ would contain an essential non-separating torus, which is false here. Suppose any essential torus is separating, and decompose $N_{p,r}$ along an outermost such torus, F, so that $N_{p,r} = A \cup_F B$ with A atoroidal and $\partial N_{p,r} \subset B$. If we assume, for a contradiction, that $N_{p,r}(-1/q)$ is small Seifert fibered for some q with $|q| > 8$, then we have that F compresses in $B(-1)$, $B(\infty)$, and $B(-1/q)$. It follows, from Lemma 2.2.1 that B is a cable space with cabling slope $\gamma = 0$. But $N_{p,r}(0)$ is neither reducible nor a lens space, so we reach a contradiction. It follows that $N_{p,r}$ is not toroidal, and must be hyperbolic.

Since $N_{p,r}$ is hyperbolic, and $N_{p,r}(\infty)$ is exceptional, it follows that for any exceptional filling $N_{p,r}(-1/q)$, $\Delta(\infty, -1/q) \leq 8$, by Theorem 2.2.10. It follows that $|q| \leq 8$, as desired. A similar argument shows that $|p| \leq 8$, as well.

$\square$

Proof of Lemma 2.3.2. We will proceed as in the lemma above by analyzing the tangle $\mathcal{S}_r = \mathcal{S}_{1,r}$ formed by removing a ball containing the q-twist region of knot $\mathcal{T}_{1,q}(r)$. We will show that the branched double cover N_r of $\mathcal{S}_r$ is hyperbolic.

Consider the following fillings on N_r (see Figure 2.3.6). Assume for a contradiction that $|q| \geq 9$ and $K_{p,q}(\alpha_r) = N_r(-1/q)$ is a small Seifert fibered space.

$$
\begin{aligned}
&N_r(-1/q) = K_{p,q}(\alpha_r)\\
&N_r(0) = S^2(1/2, -1/2, 1/(2-r))\\
&N_r(\infty) = S^2(1/2, -2/5, -1/(r+2))\\
&N_r(-1) = S^2(1/3, -1/4, -1/r)\\
&N_r(-1/2) = S^2(-1/3, 2/5, -1/(r-1))\\
&N_r(1) = K[-2/7](\alpha_r)
\end{aligned}
$$

It is clear from this that N_r is irreducible (again, by [GL89]), since it has distinct irreducible fillings, for any value of r. Suppose that N_r is Seifert fibered. Since, for all values of r, N_r has fillings that are Seifert fibered with base surface S^2, but do not contain a pair of exceptional fibers of multiplicity 2, the base surface of N_r is orientable, i.e., D^2. A Seifert fibered space with connected boundary with a small Seifert fibered filling must have 2 or 3 exceptional fibers. Furthermore, since no slope is distance one from 0, ∞, and -1, N_r has 2 exceptional fibers, i.e., $N_r = D^2(a, b)$.

By the classification of exceptional surgeries on 2-bridge knots [BW01], $N_r(1)$ is exceptional if and only if $\alpha_r \in \{0, 1, 2, 3, 4\}$. In this case, $p = 1$ and

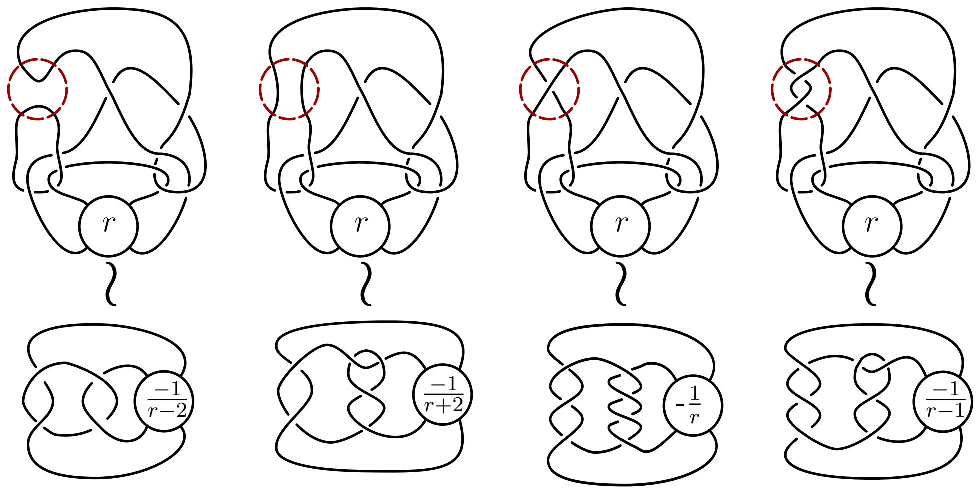


Figure 2.3.6: Four fillings, 0, ∞, 1, and 1/2, of the tangle $\mathcal{S}_r$ that help to prove that N_r is hyperbolic if $r \neq 0, 1, 2$.

$q = -1$, so $\alpha_r = 4 - r$ and this is the equivalent to $r \in \{4, 3, 2, 1, 0\}$. However, we already know that if $r = 0, 1, 2$, then $N_r(-1/q)$ is exceptional, so we only need to consider $r = 3, 4$.

If $r = 3$, then, by considering $N_3(\infty)$, we see that $a = 5$, and by considering $N_3(-1)$, we see that a cannot be 5. If $r = 4$, then by considering $N_4(-1/2)$, we see that $a = 3$, and by considering $N_4(\infty)$, we see this is impossible. It follows that N_r cannot be Seifert fibered if $r \notin \{0, 1, 2\}$.

It follows that N_r is non-Seifert fibered and, thereby, ∂-irreducible. If N_r were toroidal, since it has atoroidal fillings at distance two, it must be a cable space, by Lemma 2.2.1. However, the only cabling slope γ that satisfies $\Delta(\gamma, -1/2) = 1$ and $\Delta(\gamma, \infty) = 1$ is $\gamma = 0$, in which case we must have $N_r(0)$ be reducible or a lens space. So, we must have $r = 3$ and $N_3(0)$ is a lens space.

Suppose $N_3 = A \cup_{T^2} B$ with A atoroidal and $\partial N_3 \subset B$.

Then, because $B(\alpha) = S^1 \times D^2$ for $\alpha \in \{-1/q, \infty, -1, -1/2, 1\}$ each of these fillings induces a filling η_α on A. Since $B(0) = (S^1 \times D^2)\#L$ for some lens space L, and since $N_3(0)$ is a lens space, it follows that $A(\eta_0) = S^3$, so A is a knot complement. By construction, A is atoroidal. If A were Seifert fibered, then, by considering $A(\eta_\infty) = N_3(\infty)$ and $A(\eta_{-1}) = N_3(-1)$ just as before, we reach a contradiction.

It follows that N_r must be hyperbolic (for $r \notin \{0, 1, 2\}$). An application of Theorem 2.2.10 gives us that $\Delta(-1/q, \infty) \leq 8$ if $N_r(-1/q)$ is a small Seifert fibered space, which proves the lemma.

$\square$

2.3.1 Completing the proof of Theorem 2.1.1

The work above leaves us with a finite list of knots and links $L_{p,q,r} = \mathcal{T}_{p,q}(r)$ whose branched double covers might be Seifert fibered. We must consider non-integral r if and only if p and q are both positive. In the event of a non-integral slope r/s, we may assume $|s| \leq 8$ by Theorem 2.2.10, since $1/0$ is an exceptional filling. By Proposition 2.2.6, we must show that each of these links is not a Montesinos link or a Seifert link.

Method 1 (see Subsection 2.2.7) can be used to show that none of the $L_{p,q,r}$ are Montesinos knots or links, though it should be noted that the Kauffman polynomial must be employed in a handful of cases, including distinguishing $L_{1,4,-1}$ from $K[1/3, 2/5, -2/5]$ and some non-integral cases, and that the HOMFLYPT polynomial must be employed to distinguish $L_{1,6,-2}$ from $K[-1/4, 1/6, 2/7]$. In other words, these pairs are not distinguished by their Alexander polynomial and Khovanov homology alone.

Now, consider when $L_{p,q,r}$ is a link. Then it is the union of the unknot and the 2-bridge knot $K[n/m]$, where $n = 2pq-p-2$ and $m = 4pq-2p-2q-3$. $K[n/m]$ is a torus knot only if $p = 1$ and $q = 3$ or 4. In the latter case, $L_{1,4,r}$ is the union of a trefoil and an unknot. If this link is to be a Seifert link, the unknotted component must lie as the core of the torus upon which the trefoil sits. However, the link just described can be distinguished from $L_{1,4,r}$ for all values of r using the Jones polynomial. Concerning $L_{1,3,r}$, exceptional surgeries on the knot $P(-2, 3, 7)$ are previously well-understood [EM97].

It only remains to show that $L_{p,q,r}$ is never a torus knot. If $p \neq 1$, this is accomplished by applying Criterion 2.2.17. For $p = 1$, Criterion 2.2.18 suffices.

2.4 The case of $(3, 3, |q_3|)$

We now turn our attention to pretzel knots $P(q_1, q_2, q_3)$ such that $|q_1| = |q_2| = 3$. The case of $P(3, 3, q_3)$, where $q_3 > 0$ was handled by Ichihara and Jong in [IJ11]. We break up the remaining cases as follows.

1. $P(3, \pm 3, -2m)$ with $m \geq 1$

2. $P(3, 3, 2m + 1)$ with $m \leq -2$

3. $P(3, -3, 2m+1)$ with $m \leq -3$ or $2 \leq m$

4. $P(3, 3, 2m, -1)$ with $2 \leq m$

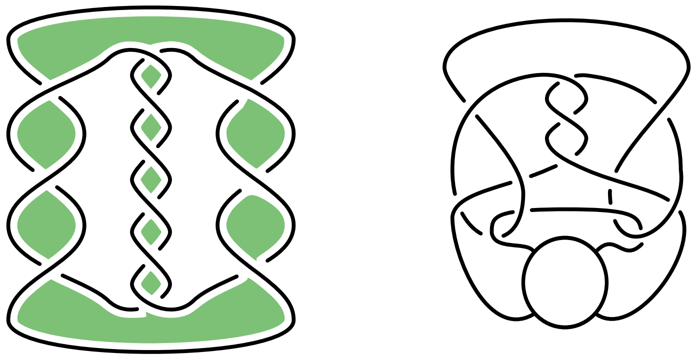
In Cases (2) and (3), it is only necessary to consider integral surgery slopes by Theorem 2.1.5. Our main result is:

Theorem F. Hyperbolic pretzel knots of the form $P(3, 3, m)$ or $P(3, 3, 2m, -1)$ admit no small Seifert fibered surgeries. Pretzel knots of the form $P(3, -3, m)$, with $m > 1$, admit small Seifert fibered surgeries precisely in the following cases:

- $P(3, -3, 2)(1) = S^2(1/3, 1/4, -3/5)$
- $P(3, -3, 3)(1) = S^2(1/2, -1/5, -2/7)$
- $P(3, -3, 4)(1) = S^2(-1/2, 1/5, 2/7)$
- $P(3, -3, 5)(1) = S^2(2/3, -1/4, -2/5)$
- $P(3, -3, 6)(1) = S^2(1/2, -2/3, 2/13)$

2.4.1 Case (1)

Let $K_m^{\pm} = P(3, \pm 3, -2m)$. To avoid the redundancy of mirrors, we can restrict to $m > 0$. Recall that these knots can only admit non-integral small Seifert fibered surgeries in the case of K_m^+. Our first result is half of Theorem F (up to mirroring).



(a) (b)

Figure 2.4.1: (a) The knot $K_m^- = P(3,-3,-2m)$ (shown here with $m = 3$) bound punctured Klein bottles. (b) The tangle $\mathcal{T}_3^+$.

Proposition 2.4.1. Let $K_m^\pm = P(3, \pm 3, -2m)$ with $m > 0$ be hyperbolic. Then, $K_m^\pm$ admits no small Seifert fibered surgeries, except in the following three instances.

- $P(3,-3,-2)(-1) = S^2(2/3, -1/4, -2/5)$
- $P(3,-3,-4)(-1) = S^2(1/2, -1/5, -2/7)$
- $P(3,-3,-6)(-1) = S^2(1/2, -1/3, -2/13)$

As in Section 2.3, we will proceed in this case by first limiting the possible surgery slopes, then limiting the size of m, then using techniques from Subsection 2.2.7 to check that small values of m and slopes satisfying the relevant bound do not produce small Seifert fibered spaces (except for the three noted cases). Let $K_m^\pm = P(3, \pm 3, -2m)$ with $m > 0$. We begin by observing that $K_m^\pm$ bounds a punctured Klein bottle. Let $\alpha_r^+ = 12 - r$ and $\alpha_r^- = -r$ (again, these are chosen so that $K_m^\pm(\alpha_r)$ corresponds to r-filling on

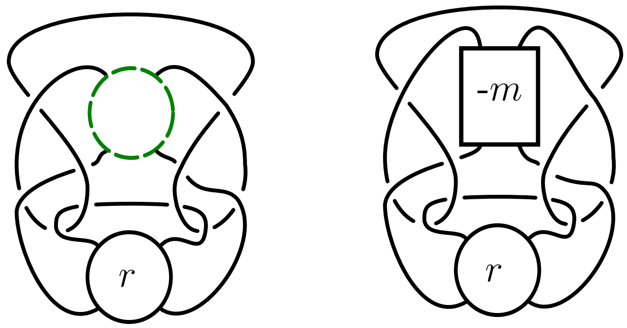


Figure 2.4.2: The tangle $\mathcal{S}_r^-$ and the link $L_{m,r}^-$

the corresponding tangle). Then this Klein bottle has boundary slope $\alpha_0^\pm$ (see Figure 2.4.1(a)). Since surgery along this slope produces a toroidal manifold (as in the previous section), any exceptional surgery slope for $K_m^\pm$ must be close to $\alpha_0^\pm$. In particular, $\Delta(\alpha_r^\pm, \alpha_0^\pm) \leq 8$.

Next, we remark that $K_m^\pm$ is strongly invertible. Let $\mathcal{T}_m^\pm$ be the resulting quotient tangle, and let $L_{m,r}^\pm = \mathcal{T}_m^\pm(r)$.

Lemma 2.4.2. Suppose $K_m^\pm(\alpha_r)$ is a small Seifert fibered space. Then, $m \leq 8$.

Proof. Let $L_{m,r}^\pm = \mathcal{T}_m^\pm(r)$ and form the tangle $\mathcal{S}_r^\pm$ by removing a 3-ball containing the m-twist box of $L_{m,r}^\pm$ (see Figure 2.4.2). Let $M_r^\pm = \widetilde{S_r^\pm}$. Of course, $\mathcal{S}_r^\pm(-1/m) = L_{m,r}^\pm$, and $K_m^\pm(\alpha_r) = M_r^\pm(1/m)$. Assume, for a contradiction, that $M_r^\pm(1/m)$ is a small Seifert fibered space for some $m \geq 9$.

Consider the following fillings of $M_r^\pm$, which we can easily visualize and verify by looking at the corresponding rational tangle fillings of $\mathcal{S}_r^\pm$ (see Figure

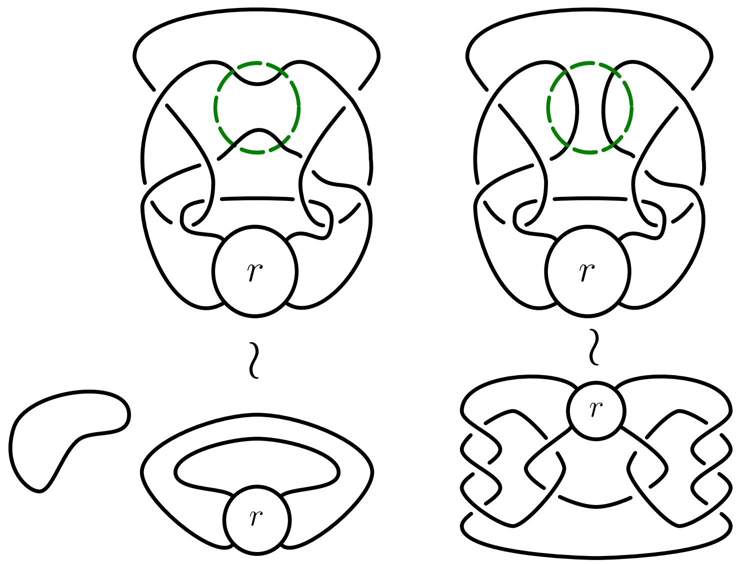


Figure 2.4.3: Two interesting fillings of $\mathcal{S}_r^-$

2.4.3).

$$\begin{aligned} M_r^\pm(1/m) &= \text{small Seifert fibered space (by assumption)} \\ M_r^-(0) &= (S^1 \times S^2)\#L(r,1) \\ M_r^\pm(\infty) &= D^2(2,3) \cup_{T^2} D^2(2,3) \end{aligned}$$

As was argued in Section 2.3, $M_r^\pm$ is irreducible (it has distinct irreducible fillings), non-Seifert fibered (it has a non-Seifert fibered, non-reducible filling), and ∂-irreducible (it is irreducible and not $S^1 \times D^2$). Assume that M_r^- is toroidal, so $M_r^- = A \cup_F B$ with A atoroidal and $\partial M_r^- \subset B$.

Suppose that F compresses in $M_r^-(\infty)$. Then, since $\Delta(1/m, \infty) = |m| \geq 2$, B is a cable space with cabling slope γ satisfying $\Delta(\gamma, \infty) = \Delta(\gamma, 1/m) = 1$. It follows that $\gamma = a \in \mathbb{Z}$, and $|ma - 1| = 1$. Since $|m| \geq 9$, a must be zero, so $\gamma = 0$. It follows that $B(0) = (S^1 \times D^2)\#L$, where L is a lens space, and $B(\infty) = B(1/m) = S^1 \times D^2$. Let η_0, η_∞, and $\eta_{1/m}$, be the slopes of the induced slopes of the meridian of B after the above fillings are performed, so $M_r^-(\alpha) = A(\eta_\alpha)$ for $\alpha = \infty$ or $1/m$, and $M_r^-(0) = A(\eta_0)\#L$. Since L has finite fundamental group, $L = L(r, 1)$ and $A(\eta_0) = S^1 \times S^2$.

Now, since $\Delta(\eta_\infty, \eta_0) \geq 4$ (by Lemma 2.2.1), A cannot be hyperbolic, because $\Delta(S^2, T^2) = 3$. Since A was assumed to be atoroidal, it follows that A must be Seifert fibered. But $A(\eta_\infty)$ is irreducible and not Seifert fibered, so this cannot be. This contradiction means that F cannot compress in $M_r^-(\infty)$.

So, assume F remains incompressible in $M_r^-(\infty)$. If $|r| = 1$, then $M_1^-(0) = S^1 \times S^2$. But since F is the unique incompressible torus in a non-Seifert fibered graph manifold, $A = D^2(2, 3)$. In particular, $M_1^-(0) = A(\eta_0) = S^1 \times S^2$ is not a possible filling of a trefoil complement.

If $|r| > 1$, then since F compresses in fillings at slopes $1/m$ and 0, which are at distance one, by Lemma 2.2.1, B is either a cable space or the exterior of a braid in a solid torus. Since $M_r^-(0) = (S^1 \times S^2)\#L(r, 1)$, either $B(0)$ or $A(\eta_0)$ is $S^1 \times S^2$. However, this is not possible for such spaces B, nor is it possible for $A = D^2(2, 3)$.

Thus, M_r^- is not toroidal, and must be hyperbolic. So, by Theorem 2.2.11, $\Delta(1/m, \infty) = |m| \leq 8$, a contradiction that yields the desired result. The reasoning is very similar to show that M_r^+ must be hyperbolic as well, noting that $M_r^+(0) = D^2(2, 2) \cup_{T^2} D^2(2, r)$ is toroidal for all r, and $M_r^+(1) = S^2(1/3, -1/4, -1/r)$.

$\square$

2.4.2 Completing the proof of Proposition 2.4.1

By our work above, we can conclude that if $P(3, \pm 3, -2m)(\alpha_r^{\pm})$ with $m > 0$ admits a small Seifert fibered surgery, then $|r|, |m| \leq 8$. Let $L_{m,r}^{\pm} = \mathcal{T}_m^{\pm}(r)$. We assume that r is integral for $L_{m,r}^{-}$.

First, consider the links $L_{m,r}^{\pm}$, which are the union of an unknotted component with a component $J_m^{\pm} = K[2/3, \pm 2/3, -1/2m]$ (to see this, consider $L_{m,0}^{\pm}$, or compare with Figure 2.4.15). Because $J_m^{\pm}$ is not a torus knot or a 2-bridge knot for any m, by Criteria 2.2.15 and 2.2.19, we can conclude that $L_{m,r}^{\pm}$ is never a 2-component Seifert link or Montesinos link.

When $L_{m,r}^{\pm}$ is a knot, we see that $2g(L_{m,r}^{\pm}) \neq s(L_{m,r}^{\pm})$, so, by Criterion 2.2.17, $L_{m,r}^{\pm}$ is never a torus knot. To see that $L_{m,r}^{\pm}$ is never a Montesinos knot, we implement Method 1 (see Subsection 2.2.7), accounting for r non-integral when necessary.

2.4.3 Case (2)

Next, we will consider hyperbolic pretzel knots of the form $K_m = P(3, 3, 2m + 1)$. Here, K_m is hyperbolic if $m \neq -1$ or 0, and if m is positive, then Ichihara-Jong have shown that K_m admits no Seifert fibered surgeries [IJ11]. The case when $m = -2$ will be covered in Subsection 2.4.5, so assume $m \leq -3$. In this section, we prove the following.

Proposition 2.4.3. A pretzel knot of the form $P(3, 3, 2m + 1)$ with $m \leq -3$ admits no small Seifert fibered surgeries.

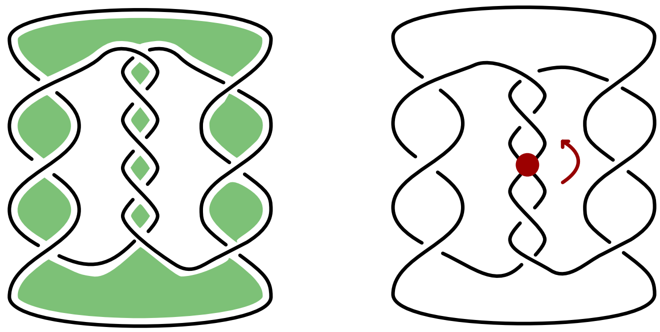


Figure 2.4.4: The left figure above shows the pretzel knot $P(3, 3, 2m + 1)$ as the boundary of a punctured torus, while the right one exhibits the strong inversion of $P(3, 3, 2m + 1)$ given by rotation. Here, $m = -3$.

As before, we will first restrict the possible values of m for which K_m might admit a Seifert fibered surgery, then rule the remaining cases out by computer. First we note that K_m has genus one, so by Theorem 2.2.12, $|r| \leq 3$ (see Figure 2.4.4). Since the three pretzel parameters are all odd, K_m cannot admit non-integral Seifert fibered surgeries by Theorem 2.1.5. Let $\alpha_r = -r$, so that $K_m(\alpha_r) = K_m(-r)$ will correspond with $\mathcal{T}_m(r)$ (the rationally-filled quotient tangle), as before.

Lemma 2.4.4. If $m \leq -10$, then $K_m(r)$ is not a small Seifert fibered space.

Proof. In a slight variation of the preceding cases, these knots possess a strong inversion that is a half-rotation of the plane. Let $\mathcal{T}_m(r)$ be the resulting quotient link, as before (see Figure 2.4.5). Again, we form the tangle $\mathcal{S}_r$ by removing a ball containing the $(m + 1)$-twist region (see Figure 2.4.6). If we denote by N_r the branched double cover of S^3 along $\mathcal{S}_r$, and assume for a contradiction that $K_m(r)$ is a small Seifert fibered space for some $m \leq -10$ and

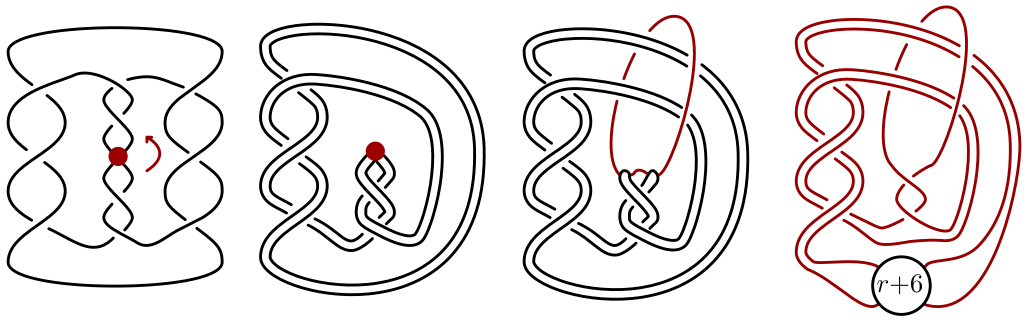


Figure 2.4.5: The figures above illustrate how to obtain the quotient tangle $\mathcal{T}_m$ by applying the Montesinos trick to the strong inversion of $P(3,3,2m+1)$ given by rotating the knot π radians through its center. Here, $m=-3$.

some r, then we have the following fillings of N_r (see Figure 2.4.6). Note that $N_r(\infty) = P(3,3,-1)(r)$, so it is simply r-surgery on the left-handed trefoil.

$$\begin{aligned} N_r(-1/(m+1)) &= \text{small Seifert fibered space (by assumption)} \\ N_r(0) &= \text{non-Seifert fibered toroidal space} \\ N_r(\infty) &= S^2(-1/2, 1/3, -1/(r+6)) \end{aligned}$$

Since N_r has distinct irreducible fillings and a non-Seifert fibered irreducible filling, N_r is irreducible, non-Seifert fibered, and ∂-irreducible. If N_r is toroidal, then since it has atoroidal fillings at distance $\Delta(-1/(m+1),\infty) = |m+1| > 2$, it has as a subspace a cable space with cabling slope $\gamma = 0$. This means that $N_r(0)$ is either reducible or a lens space, which is a contradiction. It follows that N_r is hyperbolic and that $\Delta(-1/(m+1),\infty) = |m+1| \le 8$, a contradiction that completes the proof.

□

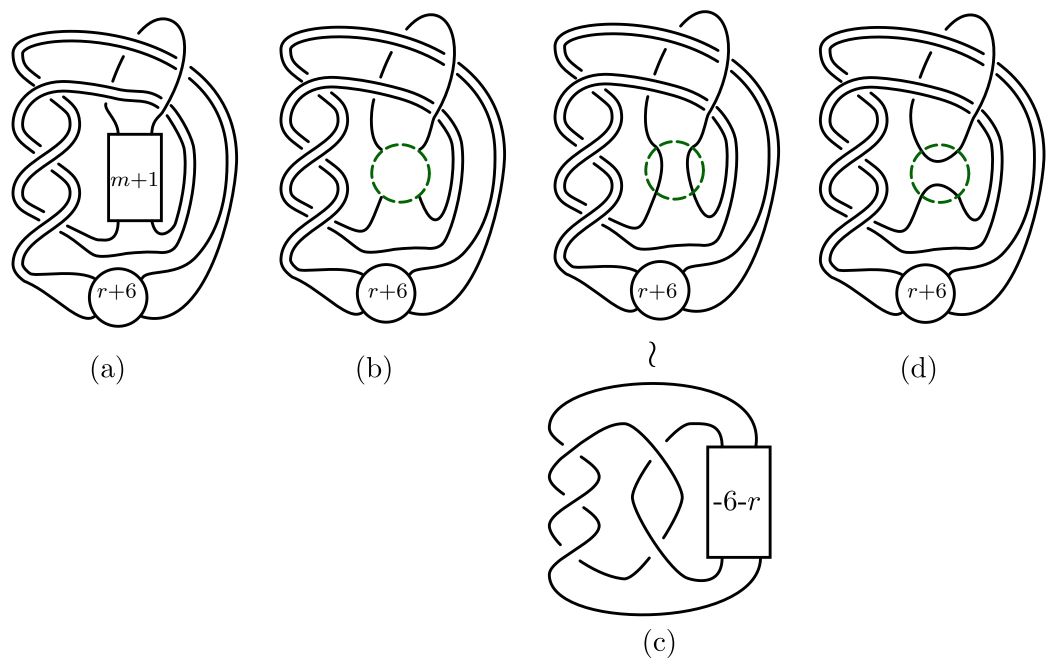


Figure 2.4.6: The first figures above, from left to right, are: (a) the link $\mathcal{T}_m(-\alpha_r)$, (b) the tangle $\mathcal{S}_r$, (c) the filling $\mathcal{S}_r(\infty)$, which is isotopic to $K[-1/2, 1/3, -1/(6+r)]$, and (d) the fillings $\mathcal{S}_r(0)$, whose link complement contains an essential torus.

2.4.4 Completing the proof of Proposition 2.4.3

Let $L_{m,r} = \mathcal{T}_m(r)$ denote the quotient link described above, and note that it is only necessary to consider $r \in \mathbb{Z}$ here. If $L_{m,r}$ is a link, then we see that it is the union of a trefoil with the knot $J_m = K[1/2, 1/3, (m-1)/(2m-3)]$. Since $m \leq -3$, by assumption, J_m is never a torus knot or a 2-bridge knot. It follows that $L_{m,r}$ is never a Seifert link or a Montesinos link, by Criteria 2.2.19 and 2.2.15, respectively.

When $L_{m,r}$ is a knot, we see that $|\mathrm{Kh}(L_{m,r})| = 6$ and $s(L_{m,r}) < \mathrm{br}(\Delta_{K_{m,r}}(t))$, so $L_{m,r}$ cannot be a Montesinos knot or a torus knot by Criteria 2.2.14 and 2.2.17, respectively.

2.4.5 Case (3)

We now consider hyperbolic pretzel knots $K_m = P(3, -3, 2m + 1)$. We will allow m to be positive or negative, which will allow us to restrict our analysis to positive surgery slopes (which must be integral if they are to be exceptional by Theorem 2.1.5). These are genus one knots, so $K_m(r)$ can only be exceptional if $|r| \leq 3$ by Theorem 2.2.12. Thus, we assume $r = 1, 2,$ or 3.

These knots are the first that we have encountered with no strong inversion (excepting $P(3, 3, -3)$), so we cannot make use of the Montesinos trick. However, K_m does have cyclic period 2, so we will study the space $K_m(r)$ by studying the link $(L_m)_f$, which is the image of $\mathrm{Fix}(f)/\langle f \rangle$ in $(K_m)_f(r/2)$ (recall this set-up from Section 2.2.5). See Figure 2.4.7.

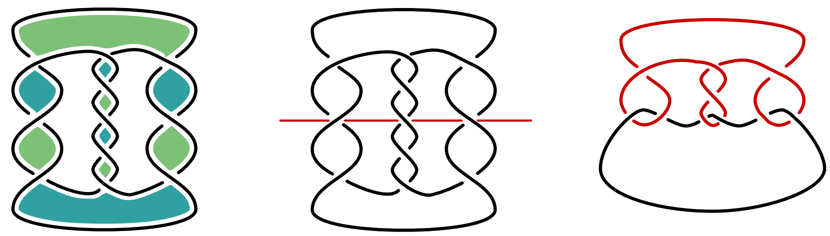


Figure 2.4.7: (a) The knot $P(3,-3,2m+1)$ (here, $m=-3$), shown as the boundary of a punctured torus, (b) along with the axis of the cycle f of period 2 of the knot, and (c) the quotient knot K_f (here, the unknot), along with the image of $\mathrm{Fix}(f)$ in the quotient.

Proposition 2.4.5. A hyperbolic pretzel knot of the form $P(3,-3,2m+1)$ admits a small Seifert fibered surgery precisely in the following instances.

- $P(3,-3,3)(1) = S^2(1/2,-1/5,-2/7)$
- $P(3,-3,5)(1) = S^2(-1/3,-1/4,3/5)$

Note that the first exceptional surgery was discovered by Song, and the second by Mattman, Miyazaki, and Motegi, see [MMM06]. Again, our first task is to restrict the possible values of m for which K_m might admit a Seifert fibered surgery, then rule out the remaining cases using knot invariants. We will handle the three cases $r = 1, 2,$ and 3 separately below.

Lemma 2.4.6. The space $P(3,-3,2m+1)(1)$ is not a small Seifert fibered space for $|m| \geq 9$.

Proof. Assume that $P(3,-3,2m+1)(1)$ is a small Seifert fibered space with $|m| \geq 9$. As we have seen $K_m(1)$ is the branched double cover of S^3 along the

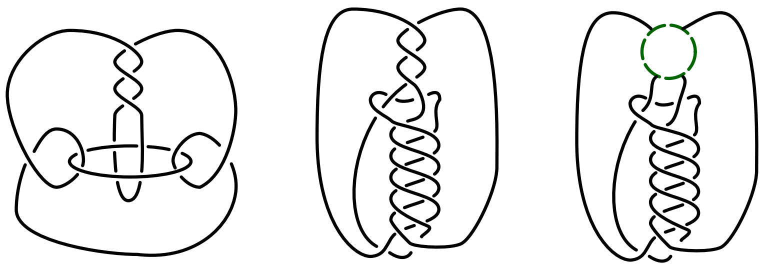


Figure 2.4.8: (a) The link $(K_m)_f \cup \mathrm{Fix}(f)$ in the quotient, (b) the knot $(L_m)_f$ resulting from (1/2)-surgery on the unknotted component, and (c) the tangle $\mathcal{S}$ formed by removing the m-twist area of the knot. Here, $m = -3$.

knot $(L_m)_f$. We form the tangle $\mathcal{S}$ by removing the m-twist box of $(L_m)_f$ (see Figure 2.4.8). Let $Z = \widetilde{\mathcal{S}}$. Then, we have the following fillings:

$$\begin{aligned}
&Z(-1/m) = \text{small Seifert fibered (by assumption)}\\
&Z(0) = S^2 \times S^1\\
&Z(-1/3) = D^2(2,3) \cup_{T^2} D^2(2,3)\\
&Z(-1/2) = S^2(3,5,7)\\
&Z(-1) = S^2(2,5,7)\\
&Z(\infty) = S^2(2,3,11)
\end{aligned}$$

It is worth noting that the last four fillings on the list correspond to exceptional fillings of hyperbolic pretzel knots. $P(3,-3,7)(1) = Z(-1/3)$, $P(3,-3,5)(1) = Z(-1/2)$, $P(3,-3,3)(1) = Z(-1)$, and $P(3,-3,1)(1) = Z(\infty)$. The lattermost is surgery on the rational knot $\mathcal{K}[-2/9]$. See Figure 2.4.9.

Because Z has distinct irreducible fillings as well as an irreducible non-Seifert fibered filling ($Z(-1/3)$ is a non-Seifert fibered graph manifold), it is impossible for Z to be Seifert fibered, reducible, or ∂-reducible. Assume that

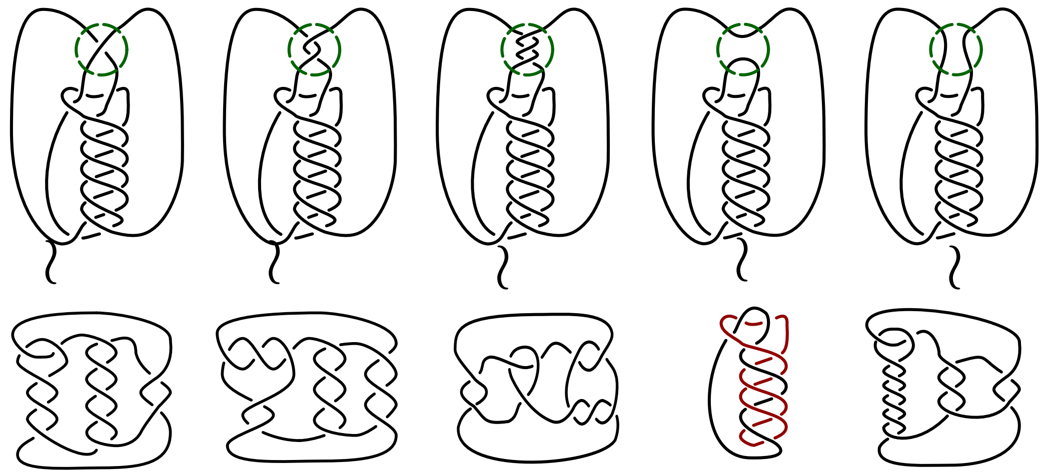


Figure 2.4.9: Above, we see five interesting fillings of $\mathcal{S}$: $\mathcal{S}(1), \mathcal{S}(1/2), \mathcal{S}(1/3), \mathcal{S}(0)$, and $\mathcal{S}(\infty)$.

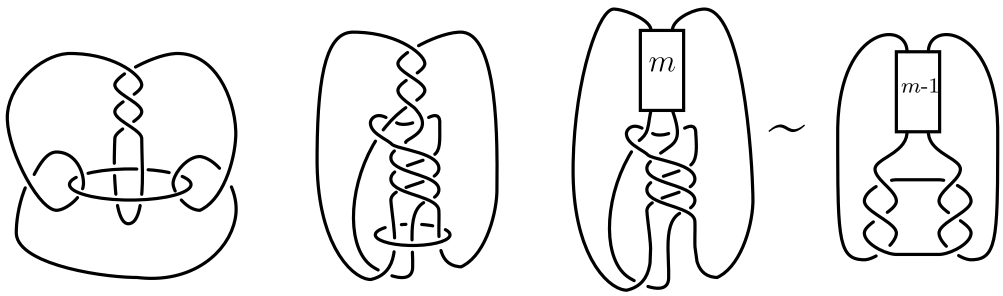
Z is toroidal, so $Z = A \cup_F B$ with A atoroidal and $\partial Z \subset B$. Then, since F compresses in $Z(-1/2)$, $Z(-1)$, and $Z(\infty)$, and $\Delta(1/2, \infty) \geq 2$, B must be a cable space. The cabling slope γ is restricted to be distance one from ∞ and $-1/2$, so $\gamma = 0$ or -1. Since $Z(-1)$ is neither a lens space nor reducible, we cannot have $\gamma = -1$. If $\gamma = 0$, then $B(0) = (S^1 \times D^2)\#(S^1 \times S^2)$, which is not a possible result of filling on a cable space.

It follows that Z is hyperbolic, so $\Delta(-1/m), \infty) \leq 8$, so $|m| \leq 8$, which gives the desired contradiction.

□

Proposition 2.4.7. $P(3, -3, 2m+1)(2)$ is never a small Seifert fibered space for $m \neq 0, -1$.

Proof. We can precede as above, by analyzing the result of 1-surgery on the



(a) (b) (c)

Figure 2.4.10: (a) The link $(K_m)_f \cup \mathrm{Fix}(f)$ in the quotient, (b) the link $(L_m)_f$ resulting from 1-surgery on the unknotted component (note that the core of the surgery torus is a component of the resulting link), and (c) the tangle formed by removing the m-twist area of the knotted component J_m of the resulting link. Here, $m = -3$.

quotient knot $(K_m)_f$. This gives a two-component link in S^3, $(L_m)_f$, such that the double cover of S^3 branched along $(L_m)_f$ is the surgery space $P(3, -3, 2m+1)(2)$. Thus, to show this surgery space is not a small Seifert fibered space, it suffices to show that $(L_m)_f$ is neither a Montesinos link of length three with two components nor a Seifert link with two components.

First, we note that $(L_m)_f$ is the union of the unknot with the knot $J_m = K[1/3, -1/3, 1/(m-1)]$ (see the right half of Figure 2.4.10). Since J_m is never torus knot, by Criterion 2.2.19, $(L_m)_f$ is never a Seifert link. If $m = 0$ or $m = 2$, then $J_0 = K[-2/9]$ and $J_2 = K[2/9]$, respectively; otherwise, J_m is not a 2-bridge knot, so $(L_m)_f$ is not a Montesinos link, by Criterion 2.2.19.

If $m = 0$, then we are considering 2-surgery on $P(3, -3, 1)$, which is $K[2/9](2)$. By the classification of Brittenham and Wu [BW01], this space is Seifert fibered. If $m = 2$, then we are considering the space $P(3, -3, 5)(2)$, and if $(L_m)_f$ is a Montesinos link, then it has the form $K[x, y, z]$, where $z = 2/9$ or $4/9$ and x and y have even denominator.

Now, since $(L_2)_f$ has a diagram with 12 crossings, and since the z–tangle would contribute 6 crossings, if $(L_2)_f$ were to be a two component length three Montesinos link, then the x– and y–tangles must contribute at most 6 crossings. Without loss of generality, we can assume $x = \pm 1/2$ and $y = \pm 1/2$ or $\pm 1/3$ or $\pm 3/4$. However, an easy check shows that the determinant of such Montesinos links cannot be 2.

□

Finally, we consider the case of 3-surgery on hyperbolic pretzel knots K_m of the type $P(3, -3, 2m + 1)$. These knots have period 2, so we can analyze the surgery space $K_m(3)$ as the double branched cover of $(K_m)_f(3/2)$, where $(K_m)_f$ is the factor knot K_m/f for the self diffeomorphism $f : S^3 \to S^3$ of order two that preserves K_m. In this case, $(K_m)_f$ is the unknot, so $(K_m)_f = -L(3, 2)$. Let L_m denote the image of $\mathrm{Fix}(f)/f$ in the surgery space $(K_m)_f(3/2)$, i.e., L_m is the branching set for the double covering. (Note that in our convention p-surgery on the unknot is the lens space $-L(p, 1)$.)

Proposition 2.4.8. $P(3, -3, 2m+1)(3)$ is never a small Seifert fibered space for $m \neq 0, -1$.

Proof. By the analysis of [MM02], the link L_m is actually a 2-bridge link, contained in the solid torus that, together with the surgery solid torus, comprises half of the genus-1 Heegaard splitting of the lens space $-L(3, 2)$ (see

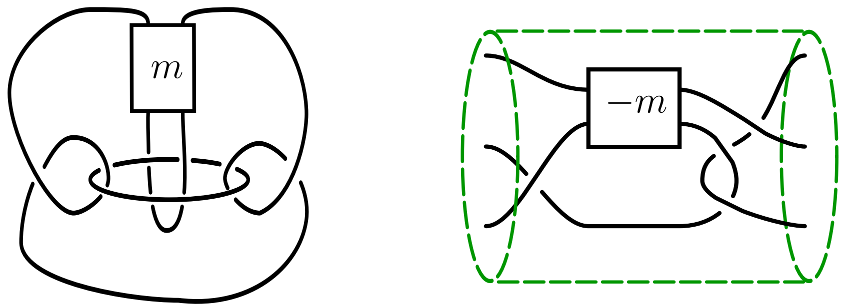


Figure 2.4.11: Above, on the left we have the link $(K_m)_f \cup \mathrm{Fix}(f)/f$ in the quotient, and, on the right, we have the result of (3/2)-surgery on the unknotted component: the link L_m contained in a solid torus (simply view the knot as lying in the solid torus that comprises the exterior of unknot).

Figure 2.4.11). Thus, if we pass to a 3-fold cover, the lift, $\widetilde{L_m}$, of L_m will be a length three Montesinos link, contained in one half of the standard genus one Heegaard splitting of S^3. We now describe how to see this.

The two solid tori that comprise the splitting of $-L(3,2)$ are attached via a map which sends the meridian of one to a $(-3/2)$–curve on the boundary of the other. Passing to the 3-fold cover changes the image of the attaching map to a $(-1/2)$–curve, which gives S^3. This lift simply triplicates the knotted part of L_m. However, if we want to think about this lift as a knot in the standard 3–sphere, we must apply a self-diffeomorphism of S^3 to get the standard Heegaard splitting of S^3 (i.e., where the meridian of one torus is glued along a $(1/0)$–curve). This final step introduces two full negative twists of the strands of L_m. The result is a link $\widetilde{L}_m$ in S^3. See Figure 2.4.12.

Since $P(3,-3,2m+1)(3)$ has quotient $-L(3,2)$, it has the form $S^2(3,3,c)$, where the exceptional fiber of multiplicity c corresponds to the branching lo-

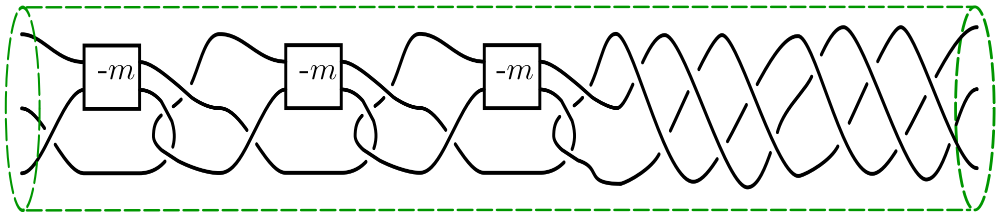


Figure 2.4.12: The link $\widetilde{L_m}$ in S^3, which is the triple cover of L_m in $-L(3,2)$.

cus, $(L_m)_f$. It follows that the lift $\widetilde{L}_m$ should be a Montesinos link of type $K[c,c,c]$. From this, it follows that c^2 divides the determinant of $\widetilde{L}_m$. We can calculate the determinant of this lift to be 49, independent of m, so it follows that $c = 7$. Thus, $P(3,-3,2m+1)(3)$ is $S^2(3,3,7)$, and $\widetilde{L}_m = K[a/7, b/7, b/7]$.

By considering the determinant (of the corresponding Montesinos knot), we see that $P(3,-3,2m+1)(3)$ must have the form $S^2(-1/3,-1/3,5/7)$ and that $\widetilde{L}_m = K[-2; 5/7, 5/7, 5/7]$ (being the triple cover of the two bridge knot $K[5/7]$ in $S^1 \times D^2$, see [MM02]). Let $V(q)$ be the Jones polynomial of $K[-2; 5/7, 5/7, 5/7]$. A straightforward calculation gives an expression for the Jones polynomial of $\widetilde{L}_m$:

$$V_{\widetilde{L}_m}(q) = \begin{cases} q^{-3m-3}(V(q) - 1) + 1 & \text{if } m \text{ is odd} \\ q^{-3m}(21 - V(q^{-1})) + 1 & \text{if } m \text{ is even} \end{cases}$$

It follows that $\widetilde{L}_m$ is not $K[-2; 5/7, 5/7, 5/7]$ unless $m = -1$. In this case the knots *are* the same, which reflects the fact that $P(3,-3,-1)(3) = S^2(3,3,7)$; however, this case is not of interest to us. For any other value of m, we have shown that $P(3,-3,2m+1)(3)$ cannot be a small Seifert fibered space. $\square$

2.4.6 Completing the proof of Proposition 2.4.5

It remains to show that the knots $L_m = \mathcal{T}_m(1)$ are neither Montesinos knots, nor Seifert knots, for $m \in [-8, 8] \backslash \{-1, 0, 1, 2\}$ (notice, when $m = 1, 2$ we *do* get small Seifert fibered surgeries, and when $m = -1, 0$, we have $P(3, -3, \pm 1)$, which are 2-bridge knots).

In fact, for these knots we have that $s(L_m) \neq 2g(L_m)$, so they cannot be torus knots by Criterion 2.2.17. Furthermore, we can apply Method 1 to show that L_m is never a Montesinos knot.

2.4.7 Case (4)

Finally, consider the case when K_m is a hyperbolic pretzel knot of the form $P(3, 3, 2m, -1)$ with $m > 1$. We note that such knots are often considered to be non-pretzel Montesinos knots. If $m = 1$, then $K_1 = P(-2, 3, 3)$, and is not hyperbolic. We see that K_m has a cyclic of period 2 (see Figure 2.4.13) with factor knot $K_f = T_{2,3}$ as well as a strong inversion. Since $K_f(r/2)$ must be a lens space surgery on the trefoil, it follows that $\Delta(r/2, 6) = 1$. In this case we consider the possibility of non-integral exceptional surgeries. We note that the link $L_{m,r} = \mathcal{T}_m(-r)$ has $4m + 10 - r$ half-twists. See Figure 2.4.15. This can be seen by carefully keeping track of the framing curve throughout the Montesinos trick.

Proposition 2.4.9. The Montesinos knots $P(3, 3, 2m, -1)$ with $m > 1$ admit no small Seifert fibered surgeries.

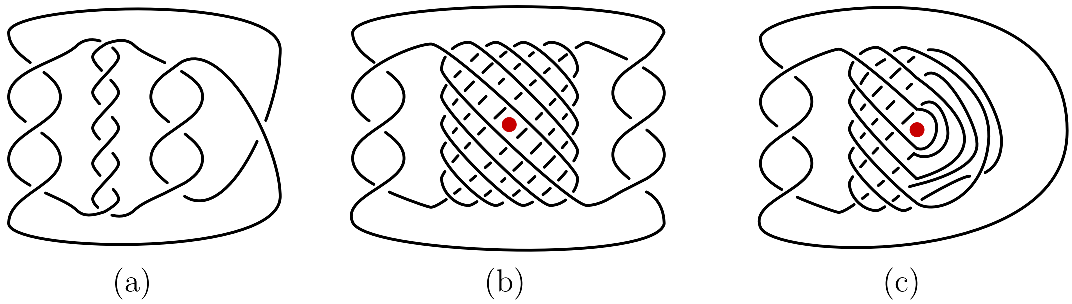


Figure 2.4.13: The Montesinos knot $P(3,3,2m,-1)$ in (a) standard form and (b) pillowcase form, and (c) the factor knot resulting from rotation about the axis perpendicular to the page.

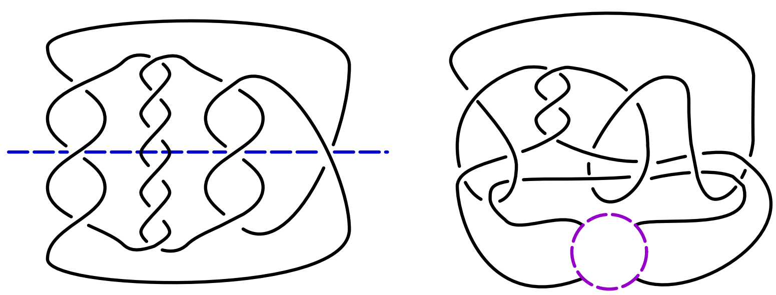


Figure 2.4.14: The Montesinos knot $P(3,3,2m,-1)$, shown with the axis of its strong inversion, and the resulting tangle $\mathcal{T}_m$. Here $m=3$.

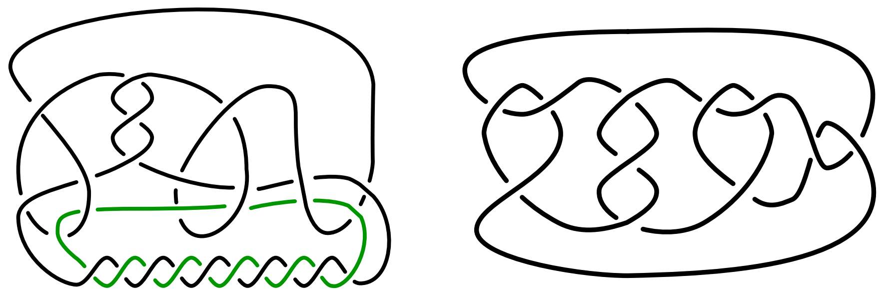


Figure 2.4.15: The link $L_{m,r}$ and the component J_m. Here $m = 3$ and $r = 12$ (hence, $4m + 10 - r = 10$ twists).

Proof. We begin by noting that we can show that $m \leq 8$ just as we did when dealing with K_m^+, earlier in this section (recall, Figure 2.4.3), so we will omit the details. Consider the quotient links $L_{m,r}$ obtained via the Montesinos trick. When $L_{m,r}$ is a link, it consists of an unknotted component, together with a component J_m, which is the knot $K[-1/3, -1/3, 1/m]$ (see Figure 2.4.15).

Since J_m is never a torus knot or a 2-bridge knot (for $m > 1$), $L_{m,r}$ is never a Seifert link or a Montesinos link, by Criteria 2.2.19 and 2.2.15, respectively.

When $L_{m,r}$ is a knot, we see that $|\mathrm{Kh}(L_{m,r})| \geq 4$ and $s(L_{m,r}) < \mathrm{br}(\Delta_{L_{m,r}}(t))$, so, by Criteria 2.2.14 and 2.2.17, $L_{m,r}$ is never a Montesinos knot or a torus knot.

$\square$

2.5 The case of $(3, 4, 5)$

We next turn our attention to the pretzel knots $P(3, \pm 4, \pm 5)$ and $P(3, 4, 5, -1)$. We will follow the same program in which we make use of the strong inversion and analyze the quotient link along with its double branched cover. Because these are not infinite families of knots, we do not need to argue to restrict any parameters as we have above. In the case of the length three pretzel knots, since these knots bound punctured Klein bottles at slope α_0, we only need to consider fillings $\alpha_r = \alpha_0 - r$ at distance at most 8 from α_0 and from $1/0$. Our only task here is to show that the quotient links are not Seifert links nor Montesinos links. In the case of $P(3, -4, 5)$, we must consider non-integral surgeries. The length four pretzel knot $P(3, 4, 5, -1)$ may also admit non-integral exceptional surgeries, and, in this case, there is no exceptional surgery by which we can bound the possible surgery slopes. On the other hand, if $P(3, 4, 5, -1)(r/s)$ is exceptional, then $|s| \leq 4$, since it is known that this pretzel knot is not tunnel number one [MSY96], and Baker, Gordon, and Luecke have recently shown that knots of tunnel number greater than one cannot have non-integral small Seifert fibered surgery slopes whose denominator is 5 or larger [BGL]. The pictures corresponding to the analysis of $P(3, 4, 5, -1)(r/s)$ are nearly identical to the diagrams in Figures 2.4.14 and 2.4.15 (which corresponded to the analysis for $P(3, 3, 2m, -1)$ in the previous section; just let $m = 2$, and change a $1/3$ tangle to a $1/5$ tangle) and the reader is encouraged to keep these in mind throughout this section.

Theorem 2.1.3. The pretzel knots $P(3, \pm 4, \pm 5)$ and $P(3, 4, 5, -1)$ admit no small Seifert fibered surgeries.

Proof. It is shown below, in Lemma 2.5.2, that the quotient links $L_{r/s}$ corresponding to the surgery spaces $P(3,4,5,-1)(r/s)$ have Khovanov homology of width at least 4 for $|s| \leq 4$ when $L_{r/s}$ is a knot. Thus, Criterion 2.2.14 suffices to prove that the knots $L_{r/s}$ are not Montesinos knots. Lemma 2.5.1 shows that $L_{r/s}$ is never a torus knot. If $L_{r/s}$ is a link, then it is the union of the unknot with the Montesinos knot $K[-2;1/2,2/3,2/5]$, which is never a 2-bridge knot or a torus knot, so $L_{r/s}$ is never a Montesinos link or a Seifert link, by Criteria 2.2.15 and 2.2.19.

Now, we consider the length three pretzel knots. Let $L_{\pm,\pm,r}$ be the quotient link resulting from the Montesinos trick, applied to $P(3,\pm 4,\pm 5)$. When $L_{\pm,\pm,r}$ is a link, $L_{\pm,\pm,r} = U \cup J$, where $J = K[2/3, \pm 1/2, \pm 2/5]$, which is never the unknot, a two-bridge knot, or a torus knot. Thus, by Criteria 2.2.15 and 2.2.19, $L_{\pm,\pm,r}$ is never a Montesinos link with two components or a Seifert link with two components.

When $L_{\pm,\pm,r}$ is a knot, $2g(L_{\pm,\pm,r}) \neq s(L_{\pm,\pm,r})$, so $L_{\pm,\pm,r}$ is not a torus knot, by Criterion 2.2.17, and we can use Method 1 to show that $L_{\pm,\pm,r}$ is never a Montesinos knot.

□

Lemma 2.5.1. $L_{r/s}$ is never a torus knot.

Proof. Recall that $|s| \leq 4$, and write $r/s = a/b + n$. A general reference for the facts in this proof is [Lic97]. If $|b| = 2$ or $|b| = 4$, then $L_{r/s}$ has unknotting number one or two. Since the unknotting number of a (p, q)–torus knot is $(p-1)(q-1)/2$, we could only have the trefoil or $T(5, 2)$. However, both of these knots are alternating, so they cannot have wide Khovanov homology, as $L_{r/s}$ will be shown to have below.

If $a/b = 0$ or $b = 3$, we can apply the oriented Skein relation to the n–twist region of $L_{a/b+n}$ to calculate a recursive formula for the Alexander polynomials $\Delta_{L_{r/s}}(t)$. In general, we write

$$\Delta_K(t) = a_0 + \sum_{i=1}^{m} a_i(t^{-i} + t^i),$$

and, applying the Skein relation to these knots, we calculate that

$$\Delta_{L_{a/b+n}}(t) = k_1(-t^{-1/2})^n + k_2(t^{1/2})^n,$$

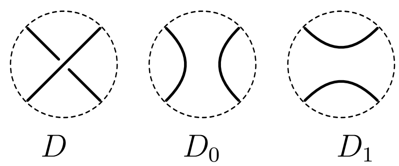
where k_1 and k_2 are fixed polynomials of small degree, depending on a/b and the sign of n. In any event, we see that a_{m-1} for $L_{a/b+n}$ will be constant as $|n|$ increases for a fixed a/b. In fact, we can calculate that $|a_{m-1}|$ takes values $3, 2, 5, 5, 7$, and 4, respectively, for the following cases: $a/b = 0$ and $n < 0$,$a/b = 0$ and $n > 0$, $a/b = 1/3$ and $n < 0$, $a/b = 1/3$ and $n > 0$, $a/b = -1/3$ and $n < 0$, and $a/b = -1/3$ and $n > 0$.

If K is a torus knot, then $|a_i| \leq 1$ for all i. It follows that $L_{r/s}$ is never a torus knot. $\square$

2.5.1 A Khovanov homology diversion

In order to prove that no surgery on $K = P(3, 4, 5, -1)$ is a small Seifert fibered space, we will argue that the quotient link, $L_{r/s}$ corresponding to $K(r/s)$ has Khovanov cohomology that is too wide when $L_{r/s}$ is a knot and $|s| \leq 4$, i.e., $|\mathrm{Kh}(L_{r/s})| \geq 4$.

We will need one important fact about Khovanov cohomology (see, for example, [Tur06], for an overview). Let D be a diagram for a knot, and let D_0 and D_1 be the diagrams identical to D, except that a single crossing has been resolved as pictured below.



Define the value c to be

$$c = \text{ (number of negative crossings in } D_0) - (\text{number of negative crossings in } D) \ .$$

Then there is a long exact sequence relating the Khovanov cohomology groups:

$$\longrightarrow \mathrm{Kh}^{i}_{j+1}(D_1) \longrightarrow \mathrm{Kh}^{i}_{j}(D) \longrightarrow \mathrm{Kh}^{i-c}_{j-3c-1}(D_0) \longrightarrow \mathrm{Kh}^{i+1}_{j+1}(D_1) \longrightarrow$$

In our examples, one of D_0 or D_1 will represent a simple knot type (unknot, Hopf link, trefoil, or (2,4)–torus link), and so the corresponding Kh will have a small range of support. Outside of this range, there will be isomorphisms between the graded components of $\mathrm{Kh}(D)$ and those of $\mathrm{Kh}(D_1)$ or $\mathrm{Kh}(D_0)$. We will make use of these isomorphism below.

Lemma 2.5.2. Let $r/s \in \mathbb{Q}$ with $r/s = a/b + t$ for $a/b \in \{0, 1/2, \pm 1/3, \pm 1/4\}$ and $t \in \mathbb{Z}$. Then $|\mathrm{Kh}(L_{r/s})| \geq 4$.

Surgery type	Kh for fixed r/s	General Kh	Diagonals $(j-2i)$	Width
Integral $(t<0)$	$\mathrm{Kh}^{0}_{1}(L_{-9}) \cong \mathbb{Q}$ $\mathrm{Kh}^{7}_{21}(L_{-9}) \cong \mathbb{Q}$	$\mathrm{Kh}^{0}_{-t-8}(L_t) \cong \mathbb{Q}$ $\mathrm{Kh}^{7}_{-t+12}(L_t) \cong \mathbb{Q}$	$\{-8-t, -2-t\}$	4
Integral $(t>0)$	$\mathrm{Kh}^{0}_{-19}(L_{11}) \cong \mathbb{Q}$ $\mathrm{Kh}^{-10}_{-31}(L_{11}) \cong \mathbb{Q}$	$\mathrm{Kh}^{0}_{-t-8}(L_t) \cong \mathbb{Q}$ $\mathrm{Kh}^{-10}_{-t-20}(L_t) \cong \mathbb{Q}$	$\{-8-t, -t\}$	5
Half-integral (t odd, $t<0$)	$\mathrm{Kh}^{-4}_{-13}(L_{1/2-5}) \cong \mathbb{Q}$ $\mathrm{Kh}^{-11}_{-21}(L_{1/2-5}) \cong \mathbb{Q}$	$\mathrm{Kh}^{t+1}_{2t-3}(L_{1/2+t}) \cong \mathbb{Q}$ $\mathrm{Kh}^{t-6}_{2t-11}(L_{1/2+t}) \cong \mathbb{Q}$	$\{-5, 1\}$	4
Half-integral (t even, $t<0$)	$\mathrm{Kh}^{-7}_{-21}(L_{1/2-6}) \cong \mathbb{Q}$ $\mathrm{Kh}^{-14}_{-29}(L_{1/2-6}) \cong \mathbb{Q}$	$\mathrm{Kh}^{t-1}_{2t-9}(L_{1/2+t}) \cong \mathbb{Q}$ $\mathrm{Kh}^{t-8}_{2t-17}(L_{1/2+t}) \cong \mathbb{Q}$	$\{-7, -1\}$	4
Half-integral (t odd, $t>0$)	$\mathrm{Kh}^{7}_{15}(L_{1/2+5}) \cong \mathbb{Q}^2$ $\mathrm{Kh}^{14}_{23}(L_{1/2+5}) \cong \mathbb{Q}$	$\mathrm{Kh}^{t+2}_{2t+1}(L_{1/2+t}) \cong \mathbb{Q}^2$ $\mathrm{Kh}^{t+7}_{2t+9}(L_{1/2+t}) \cong \mathbb{Q}$	$\{-5, 1\}$	4
Half-integral (t even, $t>0$)	$\mathrm{Kh}^{4}_{7}(L_{1/2+6}) \cong \mathbb{Q}$ $\mathrm{Kh}^{11}_{15}(L_{1/2+6}) \cong \mathbb{Q}$	$\mathrm{Kh}^{t-2}_{2t-5}(L_{1/2+t}) \cong \mathbb{Q}$ $\mathrm{Kh}^{t+5}_{2t+3}(L_{1/2+t}) \cong \mathbb{Q}$	$\{-7, -1\}$	4
Third-integral $(t<0)$	$\mathrm{Kh}^{-4}_{-9}(L_{1/3-8}) \cong \mathbb{Q}$ $\mathrm{Kh}^{-11}_{-17}(L_{1/3-8}) \cong \mathbb{Q}$ $\mathrm{Kh}^{1}_{1}(L_{-1/3-6}) \cong \mathbb{Q}$ $\mathrm{Kh}^{-9}_{-13}(L_{-1/3-6}) \cong \mathbb{Q}$	$\mathrm{Kh}^{-4}_{-t-17}(L_{1/3+t}) \cong \mathbb{Q}$ $\mathrm{Kh}^{-11}_{-t-25}(L_{1/3+t}) \cong \mathbb{Q}$ $\mathrm{Kh}^{1}_{-t-5}(L_{-1/3+t}) \cong \mathbb{Q}$ $\mathrm{Kh}^{-9}_{-t-19}(L_{-1/3+t}) \cong \mathbb{Q}$	$\{-t-9, -t-3\}$ $\{-t-7, -t-1\}$	4 4
Third-integral $(t>0)$	$\mathrm{Kh}^{1}_{-15}(L_{1/3+8}) \cong \mathbb{Q}^2$ $\mathrm{Kh}^{-8}_{-25}(L_{1/3+8}) \cong \mathbb{Q}$ $\mathrm{Kh}^{1}_{-13}(L_{-1/3-8}) \cong \mathbb{Q}^2$ $\mathrm{Kh}^{-6}_{-19}(L_{-1/3-8}) \cong \mathbb{Q}$	$\mathrm{Kh}^{1}_{-t-7}(L_{1/3+t}) \cong \mathbb{Q}^2$ $\mathrm{Kh}^{-8}_{-t-17}(L_{1/3+t}) \cong \mathbb{Q}$ $\mathrm{Kh}^{1}_{-t-7}(L_{-1/3+t}) \cong \mathbb{Q}^2$ $\mathrm{Kh}^{-6}_{-t-11}(L_{-1/3+t}) \cong \mathbb{Q}$	$\{-t-9, -t-1\}$ $\{-t-9, -t+1\}$	5 5
Fourth-Integral (t odd, $t<0$)	$\mathrm{Kh}^{-8}_{-19}(L_{1/4-9}) \cong \mathbb{Q}$ $\mathrm{Kh}^{-9}_{-15}(L_{1/4-9}) \cong \mathbb{Q}^7$ $\mathrm{Kh}^{-9}_{-27}(L_{-1/4-9}) \cong \mathbb{Q}^2$ $\mathrm{Kh}^{-10}_{-23}(L_{-1/4-9}) \cong \mathbb{Q}^2$	$\mathrm{Kh}^{t+1}_{2t-1}(L_{1/4+t}) \cong \mathbb{Q}$ $\mathrm{Kh}^{t}_{2t+3}(L_{1/4+t}) \cong \mathbb{Q}^7$ $\mathrm{Kh}^{t}_{2t-9}(L_{-1/4-t}) \cong \mathbb{Q}^2$ $\mathrm{Kh}^{t-1}_{2t-5}(L_{-1/4-t}) \cong \mathbb{Q}^2$	$\{-3, 3\}$ $\{-9, -3\}$	4 4
Fourth-Integral (t even, $t<0$)	$\mathrm{Kh}^{-11}_{-29}(L_{1/4-8}) \cong \mathbb{Q}$ $\mathrm{Kh}^{-12}_{-25}(L_{1/4-8}) \cong \mathbb{Q}^7$ $\mathrm{Kh}^{-7}_{-19}(L_{-1/4-8}) \cong \mathbb{Q}$ $\mathrm{Kh}^{-8}_{-15}(L_{-1/4-8}) \cong \mathbb{Q}^2$	$\mathrm{Kh}^{t-3}_{2t-13}(L_{1/4+t}) \cong \mathbb{Q}$ $\mathrm{Kh}^{t-4}_{2t-9}(L_{1/4+t}) \cong \mathbb{Q}^7$ $\mathrm{Kh}^{t+1}_{2t-3}(L_{-1/4-t}) \cong \mathbb{Q}$ $\mathrm{Kh}^{t}_{2t+1}(L_{-1/4-t}) \cong \mathbb{Q}^2$	$\{-7, -1\}$ $\{-5, 1\}$	4 4
Fourth-Integral (t odd, $t>0$)	$\mathrm{Kh}^{8}_{19}(L_{1/4+7}) \cong \mathbb{Q}^2$ $\mathrm{Kh}^{9}_{15}(L_{1/4+7}) \cong \mathbb{Q}$ $\mathrm{Kh}^{6}_{9}(L_{-1/4+9}) \cong \mathbb{Q}^2$ $\mathrm{Kh}^{9}_{9}(L_{-1/4+9}) \cong \mathbb{Q}^2$	$\mathrm{Kh}^{t}_{2t+5}(L_{1/4+t}) \cong \mathbb{Q}^2$ $\mathrm{Kh}^{t+1}_{2t+1}(L_{1/4+t}) \cong \mathbb{Q}$ $\mathrm{Kh}^{t-3}_{2t-9}(L_{-1/4+t}) \cong \mathbb{Q}^2$ $\mathrm{Kh}^{t}_{2t-9}(L_{-1/4+t}) \cong \mathbb{Q}^2$	$\{-1, 5\}$ $\{-3, -9\}$	4 4
Fourth-Integral (t even, $t>0$)	$\mathrm{Kh}^{6}_{11}(L_{1/4+8}) \cong \mathbb{Q}^2$ $\mathrm{Kh}^{7}_{7}(L_{1/4+8}) \cong \mathbb{Q}^2$ $\mathrm{Kh}^{7}_{15}(L_{-1/4+8}) \cong \mathbb{Q}$ $\mathrm{Kh}^{8}_{11}(L_{-1/4+8}) \cong \mathbb{Q}$	$\mathrm{Kh}^{t-2}_{2t-5}(L_{1/4+t}) \cong \mathbb{Q}^2$ $\mathrm{Kh}^{t-1}_{2t-9}(L_{1/4+t}) \cong \mathbb{Q}^2$ $\mathrm{Kh}^{t-1}_{2t-1}(L_{-1/4+t}) \cong \mathbb{Q}$ $\mathrm{Kh}^{t}_{2t-5}(L_{-1/4+t}) \cong \mathbb{Q}$	$\{-7, -1\}$ $\{-5, 1\}$	4 4

Table 2.1: For each type of a/b (first column), $\mathrm{Kh}(L_{a/b+t_0})$ can be seen to have width at least 4 (columns 2, 4, and 5). Furthermore, due to the isomorphisms exhibited in Lemma 2.5.2, these graded components persist (up to consistent grading shifts) for all $|t| > |t_0|$ (column 3), which proves that $\mathrm{Kh}(L_{a/b+t})$ always has width at least 4 (columns 3,4, and 5).

Proof. This proof will be split into cases based on the value of a/b. Values of t will be chosen so that $L_{a/b+t}$ is a knot, since the case of a link has already been covered above. Throughout, keep in mind that the diagram D of $L_{r/s}$ is a slight variation on the left side of Figure 2.4.15, as mentioned before.

First, assume that $a/b = 0$. If $t > 0$, then choose one of the negative crossings in the t-twist area of a diagram D for L_t. If $t < 0$, form D by creating a pair of opposite crossings next to the t-twist area, so that it contains a negative crossing and $t+1$ positive crossings. In either case, the 0-resolution of the negative crossing, D_0, is the unknot and the 1-resolution, D_1, is L_{t-1}. In either case, $c = -t-2$. Repeat the process once again, using D_1 as D'. Once again, D'_0 is the unknot, but now $D'_1 = L_{t-2}$ and $c' = -t-1$. Combining all of this, we have

$$\mathrm{Kh}^i_j(L_t) \cong \mathrm{Kh}^i_{j-2}(L_{t+2}) \qquad \text{if } i \neq -t-3, -t-2, -t-1.$$

Now, if we refer to Table 2.5.1, the second column provides examples of graded components of $\mathrm{Kh}(L_{11})$ and $\mathrm{Kh}(L_{-9})$ that demonstrate that these knots have wide Khovanov cohomology. But as $|t|$ increases, these graded components are preserved isomorphically (with a grading shift) in $\mathrm{Kh}(L_t)$. It follows that for large values of $|t|$, $\mathrm{Kh}(L_t)$ is also wide. For small values of $|t|$ (say, $|t| \leq 11$), it is easily verified by computer that $\mathrm{Kh}(L_t)$ is wide.

When $a/b = 1/2$, an identical argument (producing c-values of 1 and -3) gives us that

$$\mathrm{Kh}^i_j(L_{1/2+t+2}) \cong \mathrm{Kh}^{i-2}_{j-4}(L_{1/2+t}) \qquad \text{if } i \notin [-3, 3].$$

One difference here is that D_1 and D_1' are the Hopf link, instead of the unknot, so Kh vanishes outside $i \in [-2, 2]$. With this in mind, the values in Table 2.5.1 give the desired width estimates.

When $a/b = \pm 1/3$, the analysis is identical to that of the case when $a/b = 0$, except that D_0 and D_0' are a trefoil, both of whose Kh is supported in $i \in [-3, 3]$, so we have

$$\mathrm{Kh}^i_j(L_{1/3+t}) \cong \mathrm{Kh}^i_{j-2}(L_{1/3+t+2}) \qquad \text{if } i \notin [-t-6, -t+2],$$

and

$$\mathrm{Kh}^i_j(L_{-1/3+t}) \cong \mathrm{Kh}^i_{j-2}(L_{-1/3+t+2}) \qquad \text{if } i \notin [-t-3, -t+5].$$

When $a/b = \pm 1/4$, the analysis is similar to that of the case when $a/b = \pm 1/3$, except that D_0 and D_0' are both either the $T(2,4)$ or $T(2,-4)$ torus link, both of whose Kh is supported in $i \in [-4, 4]$, so we have

$$\mathrm{Kh}^i_j(L_{\pm 1/4+t}) \cong \mathrm{Kh}^{i+2}_{j+4}(L_{\pm 1/4+t+2}) \qquad \text{if } i \notin [-5, 5].$$

All these cases are concluded by regarding the information in Table 2.5.1, as has been done above.

□

2.6 Non-pretzel Montesinos knots

In this section, we discuss small Seifert fibered surgery on non-pretzel Montesinos knots. By Wu [Wu10, Wu11c], we need only consider a handful of cases: $K[1/3, -2/3, 2/5]$, $K[-1/2, 1/3, 2/(2a+1)]$ for $a \in \{3, 4, 5, 6, \}$, and $K[-1/2, 2/5, 1/(2q+1)]$ for $q \geq 1$. We prove the following theorem.

Theorem 2.1.4. Suppose that K is a non-pretzel Montesinos knot and $K(\alpha)$ is a small Seifert fibered space. Then either $K = K[-1/2, 2/5, 1/(2q+1)]$ for $q \geq 5$, or K is on the following list and has the described surgeries.

- $K[1/3, -2/3, 2/5](-5) = S^2(1/4, 2/5, -3/5)$
- $K[-1/2, 1/3, 2/7](-1) = S^2(1/3, 1/4, -4/7)$
- $K[-1/2, 1/3, 2/7](0) = S^2(1/2, 3/10, -4/5)$
- $K[-1/2, 1/3, 2/7](1) = S^2(1/2, 1/3, -16/19)$
- $K[-1/2, 1/3, 2/9](2) = S^2(1/2, -1/3, -3/20)$
- $K[-1/2, 1/3, 2/9](3) = S^2(1/2, -1/5, -3/11)$
- $K[-1/2, 1/3, 2/9](4) = S^2(-1/4, 2/3, -3/8)$
- $K[-1/2, 1/3, 2/11](-2) = S^2(-2/3, 2/5, 2/7)$
- $K[-1/2, 1/3, 2/11](-1) = S^2(-1/2, -2/7, 2/9)$
- $K[-1/2, 1/3, 2/5](3) = S^2(1/2, -1/3, -2/15)$
- $K[-1/2, 1/3, 2/5](4) = S^2(1/2, -1/6, -2/7)$
- $K[-1/2, 1/3, 2/5](5) = S^2(-1/3, -1/5, 3/5)$

- $K[-1/2, 1/5, 2/5](7) = S^2(1/2, -1/5, -2/9)$

- $K[-1/2, 1/5, 2/5](8) = S^2(-1/4, 3/4, -2/5)$

- $K[-1/2, 1/7, 2/5](11) = S^2(-1/3, 3/4, -2/7)$

Question ??. Do the Montesinos knots $K[-1/2, 2/5, 1/(2q+1)]$ with $q \geq 5$ admit small Seifert fibered surgeries?

With the exception of the case noted in the question above, we now prove the list give in Theorem 2.1.4 is complete. In proving the theorem, we will consider the three types of Montesinos knots involved separately. Note that throughout, we assume $r \in \mathbb{Z}$.

2.6.1 The case of $K[1/3, -2/3, 2/5]$

First, consider the case when $K = K[1/3, -2/3, 2/5]$. By Wu [Wu11a], $K(-4)$ and $K(-6)$ are toroidal, so it suffices to consider $K(r)$ for $-12 \leq r \leq 2$, by Theorem 2.2.10. Define L_r, as we have done before (see Figure 2.6.1, left side). To show that L_r is not a Montesinos knot or link, we apply Method 1. In the case of even r, we note that $L_r = U \cup T(2,3)$, so if L_r is a length three Montesinos link with two components, then one tangle is either a $(1/3)$–tangle or a $(2/3)$–tangle. When such a check is performed, precisely one match is found: $K[1/3, -2/3, 2/5](-5)$ is a Seifert fibered space, as shown in Theorem 2.1.4.

To see that L_r is never a Seifert link, we simply note that for odd r, L_r cannot be a torus knot, by Criterion 2.2.17, since $|s(L_r)| < \mathrm{br}(\Delta_{L_r}(q))$. If r is even, we note that $L_r = U \cup T(2,3)$. *A priori*, L_r may be a trefoil union one of its core curves; however, this link has crossing number 7, and the crossing number of L_r is at least $\mathrm{br}(V_{L_r}) = 10$. Thus, L_r is never a two component Seifert link.

2.6.2 The case of $K[-1/2, 1/3, 2/(2a+1)]$

In the case where $K_a = K[-1/2, 1/3, 2/(2a+1)]$ for $a \in \{3, 4, 5, 6\}$, we note that by Wu [Wu11a] we have the following toroidal fillings: $K_3(-2)$, $K_4(5)$, $K_5(2)$, $K_5(-3)$, and $K_6(2)$, so we consider surgery slopes r with distance at most 8 from the corresponding toroidal filling.

We proceed as above, considering links and knots $L_{a,r}$ (see Figure 2.6.1, right side), to show that $L_{a,r}$ is never a Montesinos knot or link, noting in this case that $L_{a,r} = U \cup J$, where J is $T(2,5)$ if $a = 3$, $T(2,3)$ if $a = 4$, and the unknot if $a = 5$ or 6. By applying Method 1, we find that the Montesinos links are precisely those stated in Theorem 2.1.4.

If r is even, and $L_{a,r}$ is a Seifert link, we must have that $L_{3,r}$ is the union of $T(2,5)$, together with a core curve, $L_{4,r}$ is the union of $T(2,3)$, together with a core curve, and $L_{5,r}$ and $L_{6,r}$ have the form $T(2,2n)$ for some n. However, $L_{5,r}$ and $L_{6,r}$ cannot have this form, since they are not alternating, a fact deduced by noticing that $|\mathrm{Kh}(L_{a,r})| \geq 3$. As above, $L_{4,r}$ cannot have the said form because it must have at least 9 crossings. Finally, we see that $L_{3,r}$ has linking number ± 1, while $T(2,5)$ has linking number ± 5 or ± 2 with its cores. Thus, $L_{a,r}$ cannot be a two component Seifert link.

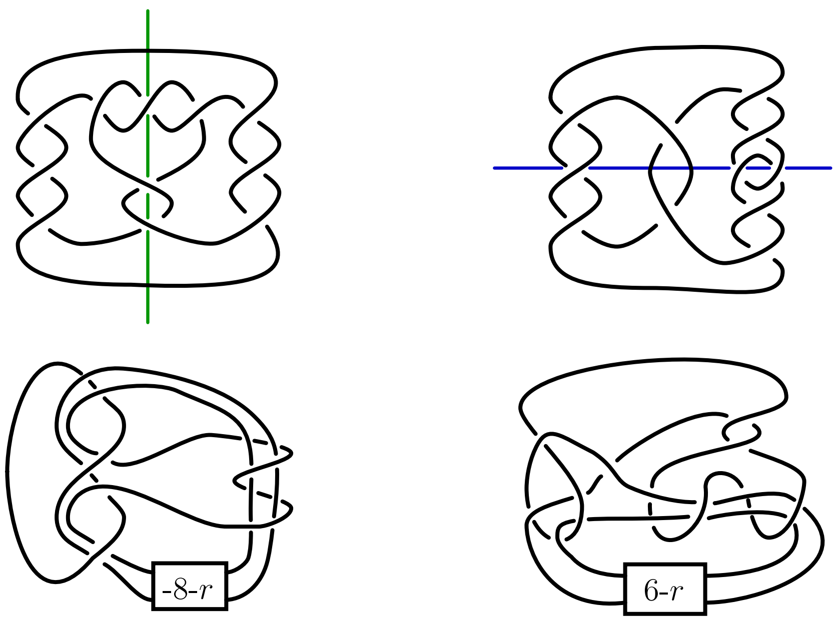


Figure 2.6.1: On the left, we have the Montesinos knot $K[1/3, 1/3 - 3/5]$, along with the axis of one of its strong inversions, and the quotient link L_r. On the right, we have the Montesinos knot $K[-1/2, 1/3, 2/9]$, along with the axis of its strong inversion, and the quotient link $L_{a,r}$ with $a = 4$. In the case of $K[-1/2, 2/5, 1/(2a+1)]$, we have a picture similar to that on the right.

If r is odd, we can obstruct most of the $L_{a,r}$ from being torus knots by using Criterion 2.2.17. However, $L_{3,1}$ and $L_{4,1}$ have equal Rasmussen invariant and breadth of Alexander polynomial. The former, in fact, corresponds to a Seifert fibered space, so consider $L_{4,1}$. Since this knot has determinant 1, if it is to be a torus knot of the form $T(a,b)$, both a and b are odd. Furthermore, since it has Rasmussen invariant equal to 10, we have $10 = (a-1)(b-1)$, which is impossible if a and b are both odd. Thus, $L_{a,r}$ is never a torus knot.

2.6.3 The case of $K[-1/2, 2/5, 1/(2q+1)]$

Finally, we are left with the case when $K_q = K[-1/2, 2/5, 1/(2q+1)]$ and $q \geq 1$. By Wu [Wu11a], we have the following toroidal fillings: $K_1(6), K_2(9), K_3(12), K_4(15)$ so for $q \leq 4$, we can bound the surgery slope r, and complete the classification. For larger q, we cannot. Furthermore, we obtain no bound on q, as we have done above. Below, we argue that when $q \leq 4$, our classification is complete.

If we again consider $L_{q,r}$, we see that for $q \leq 4$ and odd r, many $L_{q,r}$ have $s(L_{q,r}) < \mathrm{br}(\Delta_{L_{q,r}}(t))$ or $\det(L_{q,r}) > s(L_{q,r}) + 1$, so they cannot be torus knots, by Criteria 2.2.17 and 2.2.18, respectively. However, $\det(L_{1,r}) = s(L_{1,r}) + 1$ for $r \in \{5, 7, 9, 11, 13\}$. So, it is possible that $L_{1,r} = T(2,r)$. However, $|\mathrm{Kh}(L_{1,r})| = 3$ for such r, so they cannot be alternating knots.

When r is even, we note that $L_{q,r} = U \cup J_q$, where J_q is the 2-bridge knot $K[2/(2q-9)]$. This knot is only a torus knot (or the unknot) if $q = 3, 4, 5$, or 6, so $L_{q,r}$ can only be a Seifert link for these values, by Criterion 2.2.19.

However, $L_{4,r}$ is never alternating, so it cannot be $T(2, 2n)$, and, $L_{3,r}$ has at least 9 crossings, so it cannot be the union of a trefoil and a core curve.

We show $L_{q,r}$ cannot be a Montesinos knot or link by applying Method 1. When r is even, we note that one tangle would be a $(2/(2q+5))$–tangle or a $((q+3)/(2q+5))$–tangle.

Chapter 3

Distinguishing topologically and smoothly doubly slice knots

3.1 Introduction

A knot K in S^3 is called *smoothly doubly slice* if there exists a smoothly embedded, unknotted 2–sphere κ in S^4 such that $\kappa \cap S^3 = K$. Analogously, K is called *topologically doubly slice* if κ is topologically locally flat. The question of which slice knots are doubly slice was first posed by Fox in 1961 [Fox62], and Zeeman showed that $K\#(-K)$ is always doubly slice [Zee65]. Work of Sumners encapsulates what was known up to about 1970 [Sum71]. In particular, he gave necessary algebraic conditions for a knot to be doubly slice and proved that 9_{46} is the only doubly slice knot up to 9 crossings. Although his proof that 9_{46} is doubly slice is (necessarily) geometric in nature, his obstruction methods are actually purely algebraic. He showed that 9_{46} is the only knot up to 9 crossings that is algebraically doubly slice. A knot K is called *algebraically doubly slice* if there exists an invertible $\mathbb{Z}$-valued matrix P such that

$$PA_KP^\tau = \begin{bmatrix} 0 & B_1 \\ B_2 & 0 \end{bmatrix},$$

where A_K is a Seifert matrix for K, and B_1 and B_2 are square matrices of equal dimension. Matrices of this form are often called *hyperbolic*, and have been studied by Levine [Lev89]. We remark that all these concepts generalize to higher dimensions (see, for example [Sum71]), but we will restrict our attention to the classical dimension.

Since the work of Sumners, there have been three major geometric developments in the theory, all in the topologically locally flat category. In what follows, we will take 'slice' and 'doubly slice' to mean 'topologically slice' and 'topologically doubly slice' and clarify the category when necessary or helpful.

First, in 1983, Gilmer-Livingston showed, using Casson-Gordon invariants, that there exist smoothly slice knots that are algebraically doubly slice, but not doubly slice [GL83]. Second, about 10 years ago, Kim [Kim06] extended the bi-filtration technology introduced by Cochran-Orr-Teichner in [COT03] to the class of topologically doubly slice knots. At the same time, Friedl [Fri04] showed that certain η–invariants coming from metabelian representations $\pi_1(M_K) \to U(k)$, where M_K denotes 0–surgery on K, can be used to obstruct double sliceness.

In this paper, the invariants used are the correction terms coming from Heegaard Floer homology (see [OS03a]). These are smooth manifold invariants, so they are well suited to distinguish the smooth and topologically locally flat categories. A second property these invariants enjoy is the fact that, while they can be used to obstruct smooth sliceness, they do not completely vanish for smoothly slice knots, as do invariants such as the signature, τ–invariant, or s–invariant. Thus, they encode enough information to distinguish smooth double sliceness and smooth sliceness. The main result of the present paper is the following.

Theorem E. There exists an infinite family of smoothly slice knots that are topologically doubly slice, but not smoothly doubly slice.

Recall that two knots K_0 and K_1 are said to be *concordant* if $K_1\#(-K_2)$ is slice (where $-K$ denotes the mirror reverse of K) or, equivalently, if there exists a properly embedded cylinder $C \subset S^3 \times I$ such that $C \cap S^3 \times \{i\} = K_i$ for $i = 0, 1$. If K_0 and K_1 are concordant, we write $K_0 \sim K_1$. Concordance can be studied in either the smooth or the topologically locally flat categories and induces (different) equivalence relations therein. Let $\mathcal{C}$ denote the set of knots in S^3 up to smooth concordance. Under connected sum, $\mathcal{C}$ inherits an abelian group structure and is called the smooth *concordance group*. Similarly, one can define the topological concordance group $\mathcal{C}^{top}$ and the algebraic concordance group $\mathcal{G}$. There exist surjective homomorphisms

$$\mathcal{C} \xrightarrow{\psi} \mathcal{C}^{top} \xrightarrow{\phi} \mathcal{G}.$$

These groups have received a large amount of attention and many interesting theorems and examples have expanded our understanding of their nature; however, there remain many open problems. For example, it is still not known whether or not $\mathcal{C}$ and $\mathcal{C}^{top}$ contain elements of finite order greater than two. On the other hand, Levine [Lev69a, Lev69b] proved that

$$\mathcal{G} \cong \mathbb{Z}^\infty \oplus \mathbb{Z}_2^\infty \oplus \mathbb{Z}_4^\infty.$$

For an excellent survey, see [Liv05].

It would be natural to define K_0 and K_1 to be doubly concordant if $K_0\#(-K_1)$ is doubly slice. However, it is not known whether this gives an equivalence relation. The issue is the following unsolved problem.

Question 3.1.1. Suppose that K is doubly slice and that $J\#K$ is doubly slice. Then, must J be doubly slice?

Without an affirmative answer to Question 3.1.1, one cannot prove transitivity of the desired equivalence relation. We say that J is *stably doubly slice* if $J\#K$ is doubly slice for some doubly slice knot K. Then, Question 3.1.1 is simply asking whether or not there exist stably doubly slice knots that are not doubly slice. Because of these difficulties, we must adopt a different definition of doubly concordant.

Recall that two knots K_0 and K_1 are concordant if and only if there exist two slice knots J_0 and J_1 such that $K_0\#J_0 = K_1\#J_1$. This follows from the more common definition of concordant by realizing that the analogue of Question 3.1.1 for slice knots is true: If K is slice and $J\#K$ is slice, then J is slice. With this in mind, we adopt the following definition.

Definition 3.1.2. Two knots K_0 and K_1 are smoothly *doubly concordant* if there exist smoothly doubly slice knots J_0 and J_1 such that $K_0\#J_0 = K_1\#J_1$. We write $K_0 \overset{\mathcal{D}}{\sim} K_1$.

It is straightforward to verify that $\overset{\mathcal{D}}{\sim}$ is an equivalence relation. We let $\mathcal{C}_{\mathcal{D}}$ denote the set of knots in S^3 modulo this relation, which inherits an abelian group structure under connected sum and is called the smooth *double concordance group*. Analogously, we can define the topological double concordance group, $\mathcal{C}_{\mathcal{D}}^{top}$, and the algebraic double concordance group, $\mathcal{G}_{\mathcal{D}}$, and we have surjective homomorphisms

$$\mathcal{C}_{\mathcal{D}} \xrightarrow{\psi_{\mathcal{D}}} \mathcal{C}_{\mathcal{D}}^{top} \xrightarrow{\phi_{\mathcal{D}}} \mathcal{G}_{\mathcal{D}}.$$

The study of these structures is complicated by Question 3.1.1. In Subsection 3.3.6 we show that if K is smoothly stably doubly slice, then the correction terms of $\Sigma_2(K)$ must vanish in the same way as when K is smoothly doubly slice. In other words, the correction terms cannot detect the difference between smoothly doubly slice and smoothly stably doubly slice. In this light, one consequence of Theorem E is that $\mathcal{T}_{\mathcal{D}} \neq 0$, where $\mathcal{T}_{\mathcal{D}} = \ker(\psi_{\mathcal{D}})$.

In [GRS08], Grigsby, Ruberman, and Strle (building on work of Jabuka and Naik [JN07]) defined invariants that can be used to obstruct a knot from having finite order in $\mathcal{C}$. After a slight modification, we show that similar invariants can be applied to $\mathcal{C}_{\mathcal{D}}$. After restricting our attention to a certain subfamily of the knots from Theorem E, we are able to show the following.

Theorem F. There is an infinitely generated subgroup, $\mathcal{S}$, inside $\mathcal{T}_{\mathcal{D}}$ generated by smoothly slice knots whose order in $\mathcal{C}_{\mathcal{D}}$ is at least three.

One would like to say that the knots in $\mathcal{S}$ have infinite order in $\mathcal{C}_{\mathcal{D}}$. Unfortunately, due to Question 3.1.1, we can only obstruct order one and order two.

Conjecture G. The subgroup $\mathcal{S} \subset \mathcal{T}_{\mathcal{D}}$ is isomorphic to $\mathbb{Z}^\infty$.

Note that an affirmative answer to Question 3.1.1 would imply Conjecture G. If the conjecture is not true, then there are knots in $\mathcal{S}$ whose branched double covers do not smoothly embed in S^4, but do stably embed in S^4 smoothly. One corollary to Theorem E is the existence of an infinite family of rational homology 3–spheres that embed in S^4 topologically, but not smoothly. Note that these manifolds are not integral homology spheres.

Organization

In Section 3.2, we give a brief outline of the proofs of Theorems E and F and give a background overview of the relevant theories. In Section 3.3, we give the construction of the pertinent family of knots and prove that they are topologically doubly slice. We also introduce and discuss the 3–manifolds and 4–dimensional cobordisms that are used in the proof of Theorem E, discuss the sub-family of knots used to prove Theorem F, and address the subtlety of Question 3.1.1. In Section 3.4 we recall the pertinent aspects of Heegaard Floer theory. In Section 3.6, we perform the calculations necessary to prove that the knots are not smoothly doubly slice. In Section 3.7, we use invariants introduced by Grigsby, Ruberman, and Strle to prove Theorem F. The proofs of the main theorems rely on calculations of the knot Floer complexes for certain torus knots and the positive, untwisted Whitehead double of the right-handed trefoil. These facts, some of which are found in [HKL12], are presented in Section 3.5.

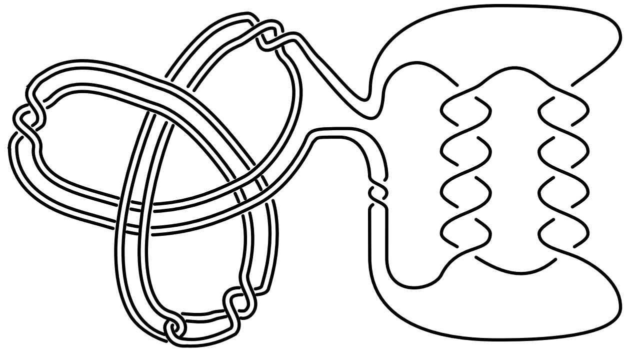


Figure 3.1.1: One member of the family $\mathcal{K}_p$; here, $p = 5$.

Acknowledgements

For the results in the present chapter, the author owes a great deal of gratitude to Çağri Karakurt and Tye Lidman, who generously shared their insight and knowledge of Heegaard Floer homology on numerous occasions and whose comments and ideas throughout this project were invaluable. The author would also like to thank his advisor, Cameron McA. Gordon, for his continued support and guidance and for freely sharing his expertise and comprehensive knowledge of all things knot theoretical. This work was supported by NSF grant number DMS-1148490.

3.2 Background and outline of proof

In Section 3.3, we construct the knots $\mathcal{K}_p$ for odd primes p, and prove that they are topologically doubly slice. (See Figure 3.1.1 for an example.)

The most difficult task of this paper is showing that the $\mathcal{K}_p$ are not smoothly doubly slice. This is accomplished by studying the double covers of

S^3 branched along these knots. If K is a smoothly doubly slice knot, then it is the intersection of a smoothly unknotted 2–sphere $\kappa \subset S^4$ with the standard $S^3 \subset S^4$. So we have $(S^3, K) \subset (S^4, \kappa)$, where the first pair sits as the equator of the second. Taking the branched double cover, we get $(\Sigma_2(K), K) \subset (S^4, \kappa)$. This gives a smooth embedding of the branched double cover $\Sigma_2(K)$ of K into S^4. We have proved the following.

Proposition 3.2.1. If K is a smoothly doubly slice knot, then $\Sigma_2(K)$ embeds smoothly into S^4.

So, we can prove that a knot is not smoothly doubly slice by showing that its branched double cover doesn't embed smoothly in S^4. To do this, we make use of the correction terms coming from Heegaard Floer homology. For more details, see Section 3.4. For now, let M denote a closed 3–manifold, and let $\mathfrak{s} \in \mathrm{Spin}^c(M)$. Let $d(M, \mathfrak{s})$ denote the correction term associated to the pair $(M, \mathfrak{s})$. The main tool in this paper is the following theorem, which also appears in [Don12] and [GL83] in one form or another.

Theorem 3.2.2. Let M be a rational homology 3–sphere that embeds smoothly in S^4. Then $H_1(M) = G_1 \oplus G_2$ with $G_1 \cong G_2$. Furthermore, there is an identification $\mathrm{Spin}^c(M) \cong H^2(M; \mathbb{Z}) \cong H_1(M)$ such that

$$d(M, \mathfrak{s}) = 0 \qquad \forall \mathfrak{s} \in G_1 \cup G_2.$$

In other words, if $|H^1(M; \mathbb{Z})| = n^2$, then at least $2n - 1$ of the n^2 correction terms associated to M must vanish.

Proof. Since M embeds smoothly in S^4, we get a decomposition $S^4 = U_1 \cup_M U_2$, where U_i is a rational homology 4–ball for $i = 1, 2$. Let $G_i = H_1(U_i)$ for $i = 1, 2$. By analyzing the Mayer-Vietoris sequence induced by this decomposition, we see that $H_1(M) \cong H_1(U_1) \oplus H_1(U_2) = G_1 \oplus G_2$. The proof that $G_1 \cong G_2$ is due to Hantzsche [Han38], and is as follows. By analyzing the relative sequence for (S^4, U_1), we see that $H_1(U_1) \cong H_2(S^4, U_1)$. By excision, $H_2(S^4, U_1) \cong H_2(U_2, M)$, and by Lefschetz duality, $H_2(U_2, M) \cong H^2(U_2)$. Finally, by the universal coefficients theorem, $H^2(U_2) \cong H_1(U_2)$ (since $H_1(U_2)$ and $H_2(U_2)$ are both torsion).

Now consider the dual isomorphism $G_1 \oplus G_2 \cong H^2(M)$, whose restrictions to G_i are induced by the inclusion $M \hookrightarrow U_i$ for $i = 1, 2$. Elements in $H^2(M)$ that are in the image of this inclusion from G_i correspond to Spin^c structures on M that extend to Spin^c structures over U_i for $i = 1, 2$. However, for any 3–manifold Y and $\mathfrak{s} \in \mathrm{Spin}^c(Y)$, we have that $d(Y, \mathfrak{s}) = 0$ if $(Y, \mathfrak{s}) = \partial(W, \mathfrak{t})$, where W is a rational homology 4–ball and $\mathfrak{t}$ extends $\mathfrak{s}$ (see [OS03a]).

If follows that $d(M, \mathfrak{s}) = 0$ for any $\mathfrak{s} \in G_1 \cup G_2$, which is a set of cardinality $2n - 1$. □

Let $\mathcal{Z}_p$ denote the double cover of S^3 branched along the knot $\mathcal{K}_p$. In Section 3.6, we make use of the surgery exact triangle to relate the Heegaard Floer homology of $\mathcal{Z}_p$ to that of simpler manifolds (manifolds obtained as surgery on knots in S^3, to be precise). Using this set-up, we show in Corollary

3.6.2 that only $2p - 3$ of the p^2 correction terms associated to $\mathcal{Z}_p$ vanish. By Theorem 3.2.2, this implies Theorem E.

Of course, the statement that at least $2n - 1$ of the n^2 correction terms must vanish does not use the full strength of Theorem 3.2.2, since it makes no use of the group structure of the correction terms. Jabuka and Naik [JN07] used this group structure to prove that many low crossing knots (whose concordance order was unknown) are not order 4 in $\mathcal{C}$. Grigsby, Ruberman, and Strle investigated this concept further [GRS08], and introduced knot invariants that can be used to obstruct finite concordance order among knots. We refine one set of these invariants so that they can be used to obstruct order one and order two in the double concordance group, and use them to prove that a family related to the $\mathcal{K}_p$ generates an infinite rank subgroup in $\mathcal{C}_{\mathcal{D}}$ (see Section 3.7). This proves Theorem F.

3.3 Geometric considerations

In this section, we use the method of infection to construct the knots $\mathcal{K}_p$ and $\mathcal{K}_{p,k}$. We then describe a sufficient condition for a knot to be doubly slice and use it to prove that these knots are topologically doubly slice. Next, we introduce the 3–manifolds triad that will be used in Section 3.6, and describe the 4–dimensional cobordisms relating them. Finally, we address Question 3.1.1.

3.3.1 Infection and the knots $\mathcal{K}_p$

Let $\vec{\eta} = (\eta_1, \ldots, \eta_n)$ be an n–component unlink in S^3, and choose an open tubular neighborhood N_i of each η_i such that $\overline{N}_i \cap \overline{N}_j = \emptyset$ for $i \neq j$. Let $E = S^3 - \cup_{i=1}^n N_i$. Next, consider a collection of knots $\vec{J} = (J_1, \ldots, J_n)$, and let E_{J_i} denote the exterior of J_i. Let M be the manifold obtained by gluing E_{J_i} to E along ∂N_i such that the meridian and longitude of η_i are identified with the longitude and meridian of J_i, respectively. This choice of gluing ensures that M is diffeomorphic to S^3.

Let $K \subset E$, and let $f : E \to M$ be the natural inclusion. Then the knot $K_{\vec{\eta}}(\vec{J}) = f(K)$ is the result of *infection* on K by $\vec{J}$ along $\vec{\eta}$. In the case when $\vec{\eta}$ is a knot, we simply write $I_\eta(J)$. See Figure 3.3.3. The term "infection" was coined in [COT03].

Example 3.3.1.

1. If $n = 1$, we recover the satellite construction. In particular, if η is chosen to be a meridian of K, then infection of K by J along η is simply $K\#J$.

2. If $K \cup \eta$ is the positive Whitehead link (see Figure 3.3.1 (b)), then infection of K by J along η is the positive, untwisted Whitehead double of J, which we denote by $Wh^+(J, 0)$. For example, if J is the right-handed trefoil, then $Wh^+(J, 0)$ is shown in Figure 3.3.1 (c).

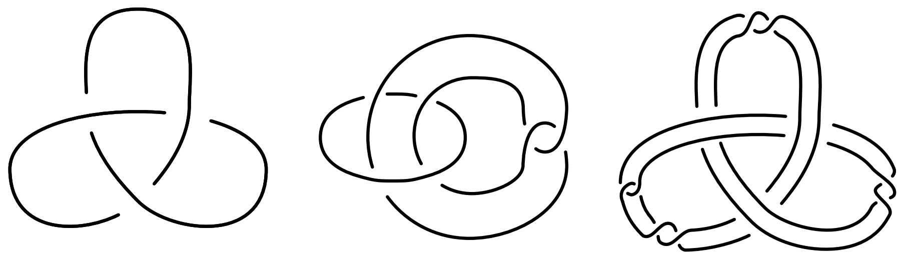


Figure 3.3.1: (a) The right-handed trefoil, (b) the positive Whitehead link, and (c) the positive, untwisted Whitehead double of the right-handed trefoil.

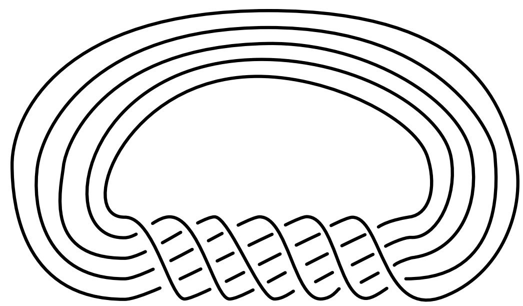


Figure 3.3.2: An example of the torus knot $T_{p,p+1}$; here $p = 5$.

Throughout, we will denote the (p, q)–torus knot by $T_{p,q}$ for $2 \leq p < |q|$ (see Figure 3.3.2).

Let $I_{J,p}$ denote the knot obtained by infecting $T_{2,p}\#(T_{2,-p})$ with J along η (see Figure 3.3.3). Let D be the positive, untwisted Whitehead double of the right handed trefoil, and let $\mathcal{K}_p = I_{D,p}$ for p an odd prime (see Figures 3.1.1 and 3.3.3(b)). Let $\mathcal{K}_{p,k} = I_{\#_k D,p}$, and note that $\mathcal{K}_{p,1} = \mathcal{K}_p$. The rest of the paper will be devoted to proving that these knots are topologically doubly slice, but not smoothly doubly slice.

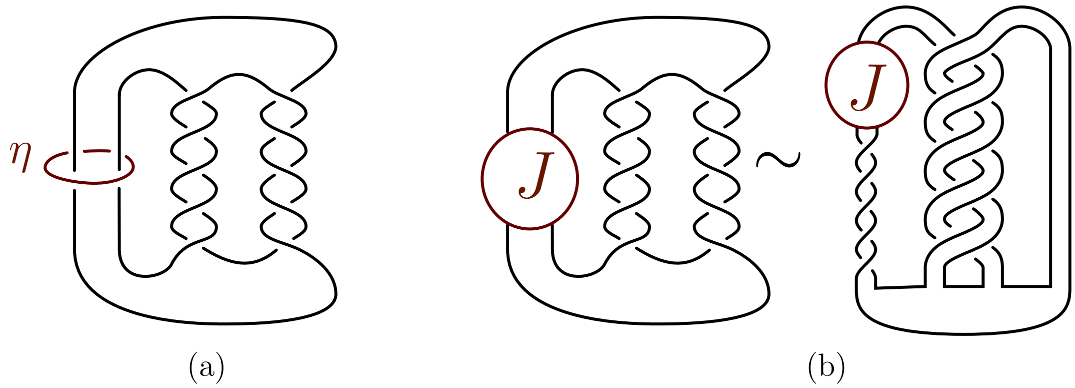


Figure 3.3.3: (a) The knot $T_{2,p}\#T_{2,-p}$ along with the infection curve η. (b) Two descriptions of the result of infecting $T_{2,p}\#T_{2,-p}$ with some knot J along η. Here, $p = 5$.

3.3.2 A sufficient condition for double sliceness

In this subsection we will present a sufficient condition for a knot K to be doubly slice that applies when K is obtained by a certain type of infection. We remark that Donald [Don12] gives a different sufficient condition: one which involves systems of ribbon bands for K.

Our criterion will make use of some well-known facts about topologically locally flat surfaces in 4–manifolds that result from the work of Freedman and Quinn [Fre82, FQ90].

Theorem 3.3.2.

1. Let K be a knot in S^3 with Alexander polynomial $\Delta_K = 1$. Then, there exists a topologically locally flat disk D properly embedded in B^4 with $\partial D = K$ and $\pi_1(B^4 - D) \cong \mathbb{Z}$.

2. Let κ be a topologically locally flat 2–knot in S^4 with $\pi_1(S^4 - \kappa) \cong \mathbb{Z}$. Then, there exists an embedded 3–ball $B \subset S^4$ with $\partial B = \kappa$.

There is a simple corollary to this theorem that will be useful below.

Corollary 3.3.3. Let K be a knot in S^3 with $\Delta_K = 1$. Then, K is topologically doubly slice.

Proof. By Theorem 3.3.2, we know that K bounds a topological disk $D \subset B^4$ whose complement has fundamental group $\mathbb{Z}$. Moreover, we have $\pi_1(S^3 - K) \to \pi_1(B^4 - D) \cong \mathbb{Z}$ is surjective. If we double the pair (B^4, D) along the boundary (S^3, K), then we get (S^4, κ), where κ is a topological 2–knot. It follows that $\pi_1(S^4 - \kappa) \cong \mathbb{Z}$ by van Kampen's theorem (this uses the surjectivity). Thus, κ is topologically unknotted with $\kappa \cap S^3 = K$, so K is topologically doubly slice. □

Proposition 3.3.4. Let K be a smoothly doubly slice knot and let $K' = I_{\vec{\eta}}(\vec{J})$ be the result of infecting K with the knots J_i, each of which is topologically doubly slice. Then K' is topologically doubly slice.

Proof. We can isotope the link $K \cup \vec{\eta}$ so that the η_i span small, disjoint disks D_i for $i = 1, \ldots, n$, which K intersects transversely in m_i points. Because K is smoothly doubly slice, there is a smoothly unknotted 2–sphere $\kappa \subset S^4$ such that $\kappa \cap (S^3 \times [-1, 1]) = K \times [-1, 1]$. From each $D_i \times I \times [-1, 1]$, we will remove the interior of a small 4–ball B_i such such that $B_i \cap (K \times [-1, 1])$ is a disjoint collection of m_i parallel disks and $B_i \cap (S^3, K)$ is a trivial tangle of m_i

strands. Let $m = \sum_{i=1}^{n} m_i$. Let $\overline{B}$ be the result of this removal, i.e., to form $\overline{B}$ we have removed n 4–balls from S^4 and and m smooth 2–disks from κ to form a punctured manifold pair.

Now, let J_i be one of the topologically doubly slice knots that will be used in the infection. Let $\mathcal{J}_i$ be a topologically unknotted 2–sphere in S^4 such that $\mathcal{J}_i \cap (S^3 \times [-1,1]) = J_i \times [-1,1]$. Let λ_i denote the disjoint union of m_i parallel copies of $\mathcal{J}_i$. Then, $\lambda_i \cap S^3$ is the $(m_i, 0)$–cable C_i of J_i, and $\lambda_i \cap (S^3 \times [-1,1]) = C_i \times [-1,1]$.

We can assume that the parallel copies of $\mathcal{J}_i$ are close enough so that there is a small 4–ball $B_i' \subset S^3 \times [-1,1]$ such that $B_i' \cap (C_i \times I)$ is a collection of m_i parallel disks and $B_i' \cap (S^3, C_i)$ is a trivial tangle of m_i strands. Form $\overline{B}_i$ by removing the interior of B_i'. Then $\overline{B}_i$ is a 4–ball that contains m_i parallel topologically unknotted disks that intersect the B^3 cross-section of B^4 in the tangle (B^3, C_i), i.e., a 3–ball containing m_i arcs that are tied in C_i.

Finally, we will reform S^4 from $\overline{B}$ by gluing $\overline{B}_i$ to $\overline{B}$ along ∂B_i to $\partial B_i \subset \overline{B}$. This has the effect of replacing each parallel set of m_i smooth disks that we removed from κ with a parallel set of m_i topological disks. Since κ was originally smoothly unknotted, this new 2–sphere κ' is clearly topologically unknotted. Furthermore, for each i, we removed from (S^3, K) a trivial tangle of m_i strands. We have now replaced that tangle with the (B^3, C_i) tangle described above. The result of this is to tie the m_i strands in the knot C_i. This is precisely the effect of infection of K with J_i along η_i. In other words, κ' is a topologically unknotted 2–sphere with $\kappa' \cap S^3 = I_{\vec{\eta}}(\vec{J}) = K'$. It follows that K' is topologically doubly slice. $\square$

We can apply the previous proposition to the knots $\mathcal{K}_{p,k}$, proving that the knots referenced in Theorems E and F are topologically doubly slice.

Corollary 3.3.5. The knots $\mathcal{K}_{p,k}$ are topologically doubly slice and smoothly slice.

Proof. Let $K = T_{2,p} \# T_{2,-p}$, let $J = \#_k D$, and let η be as shown in Figure 3.3.3. Then, $\mathcal{K}_{p,k} = K_\eta(J)$, and K is smoothly doubly slice (by Zeeman [Zee65]) and J is topologically doubly slice (by Corollary 3.3.3, since $\Delta_J = 1$). Thus, by Proposition 3.3.4, $\mathcal{K}_{p,k}$ is topologically doubly slice.

To see that $\mathcal{K}_{p,k}$ is smoothly slice, consider it as the boundary of a punctured Klein bottle, as in Figure 3.3.4. This punctured Klein bottle is formed by attaching two bands to a disk. In this case, the right most band is unknotted and untwisted. It follows that we can push the interior of the punctured Klein bottle into the 4–ball and surger it along the core of this band. The result is a smooth, properly embedded disk in the 4–ball with boundary $\mathcal{K}_{p,k}$. $\square$

3.3.3 Relevant 3–manifolds and 4–dimensional cobordisms

Let $I_{J,n}$ be the infected knot described above, and let $Z_{J,n}$ be the double-cover of S^3 branched along $I_{J,n}$. In [AK80], Akbulut and Kirby described how to get a surgery diagram for the double-cover of B^4 branched

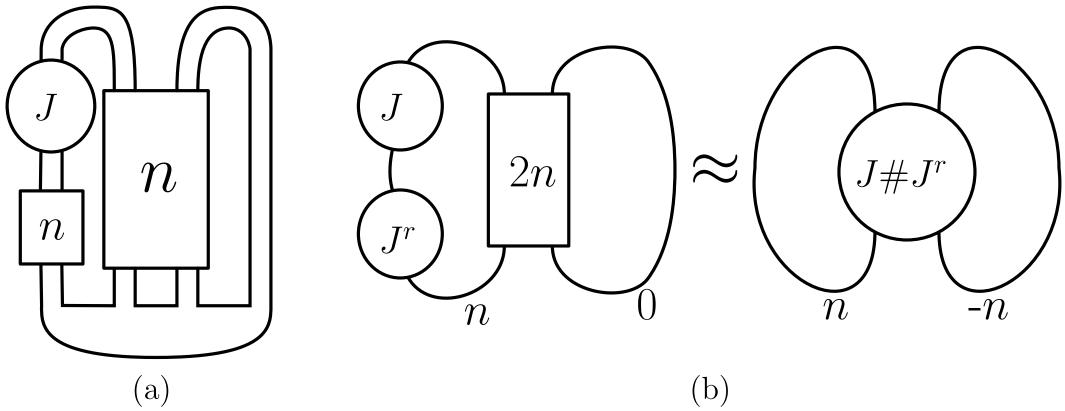


Figure 3.3.4: (a) The knot $I_{J,n}$, shown as the boundary of a punctured Klein bottle. The boxes indicate n positive half-twists. (b) Two descriptions of the resulting branched double cover, $Z_{J,n}$, which are related by a handleslide.

along a surface bounded by a knot. Applying this technique, we see that $Z_{J,n} = S^3_{n,-n}((J\#J)_{(2,0)})$, i.e., surgery on the (2,0)–cable of $J\#J$ with surgery coefficients n and $-n$ (see Figure 3.3.4). Note that throughout this paper, J will be a reversible knot, so $J^r = J$.

Let $X = S^3_n(J\#J)$, and let $K \subset X$ be the null-homologous knot shown in Figure 3.3.5. If we think of X as n–surgery on one component of the (2,0)–cable of $J\#J$, then K is the image (in the surgery manifold) of the second component of the (2,0)–cable. Since K is a longitudinal push-off of $J\#J$ in S^3, it bounds, in S^3, a Seifert surface F with $g(F) = g(J\#J)$. Since F is disjoint from $J\#J$, we see that F is a Seifert surface for K in X, as well. Thus, K is null-homologous in X. With respect to the Seifert framing of K in X, we have $X_{-n}(K) = Z_{J,n}$.

Now, let $Y = X_{-n-1}(K)$. After performing a handle-slide and blowing

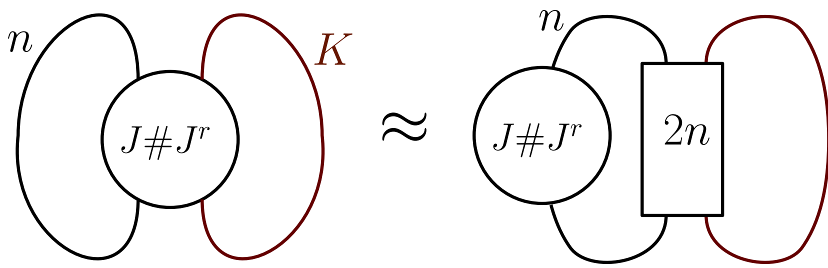


Figure 3.3.5: Two equivalent views of the null-homologous knot K in $X = S^3_n(J\#J^r)$. Note that the Seifert framing on K is different in these two descriptions. Compare with Figure 3.3.4 to see that Z is obtained by surgery on K.

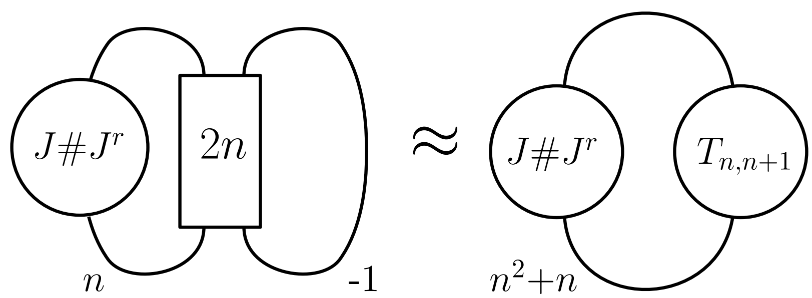


Figure 3.3.6: The manifold Y is obtained as (-1)–surgery on K in X. After a blowdown, Y can be realized by (n^2+n)–surgery on $J\#J\#T_{n,n+1}$.

down (see Figure 3.3.6), we see that $Y = S^3_{n^2+n}(J\#J\#T_{n,n+1})$. These three manifolds, X, Y, and $Z = Z_{J,n}$ form a triad:

$$\begin{tikzcd} X \arrow[dr, "W_1"'] & & Z \arrow[ll, "W_3"'] \\ & Y \arrow[ur, "W_2"'] & \end{tikzcd}$$

Now, since $-\overline{W_3}$ is the cobordism from X to Z corresponding to attaching a $(-n)$–framed 2–handle along K in X, we have that $H_2(-\overline{W_3}) \cong \mathbb{Z}$ is generated by the class $S_3 = F \cup D^2$, i.e., the genus g Seifert surface for K, capped off with the core disk of the 2–handle, and $[S_3] \cdot [S_3] = -n$ in $-\overline{W_3}$. Therefore, W_3 is a positive definite cobordism whose second homology is generated by a surface of genus $g(J \# J)$ with self-intersection n.

Similarly, W_1 is formed by attaching a $(-n-1)$–framed 2-handle to X along K. The result is that W_1 is a negative definite cobordism whose second homology is generated by a class $[S_1]$, where S_1 is a surface of genus $g = g(K)$ with self-intersection $-n-1$. Note also that $H^2(W_1) \cong \mathbb{Z}_n \oplus \mathbb{Z}$. The map from $H^2(W_1) \to H^2(X)$ induced by restricting to X is realized by projection onto the first component: $\mathbb{Z}_n \oplus \mathbb{Z} \to \mathbb{Z}_n$, while the corresponding map from $H^2(W_1) \to H^2(Y)$ is reduction modulo $n+1$ of the second component: $\mathbb{Z}_n \oplus \mathbb{Z} \to \mathbb{Z}_{n+1}$.

Finally, W_2 is obtained by attaching a (-1)–framed 2-handle along the meridian μ shown in Figure 3.3.7. In fact, μ is rationally null-homologous, and bounds a rational Seifert surface, S_2. It turns out that this surface has self-intersection $-n^2 - n$ and $[S_2]$ generates the second homology of W_2, so W_2 is negative definite. Note also that $H^2(W_2) \cong \mathbb{Z}_n \oplus \mathbb{Z}$. The map from $H^2(W_2) \to H^2(Y)$ induced by restricting to Y is realized by reduction modulo $n+1$ of the second component and the identity on the first: $\mathbb{Z}_n \oplus \mathbb{Z} \to \mathbb{Z}_n \oplus \mathbb{Z}_{n+1}$, while the corresponding map from $H^2(W_2) \to H^2(Z)$ is reduction modulo n of the second component and the identity on the first: $\mathbb{Z}_n \oplus \mathbb{Z} \to \mathbb{Z}_n \oplus \mathbb{Z}_n$.

Let's see why the capped off rational Seifert surface has self-intersection

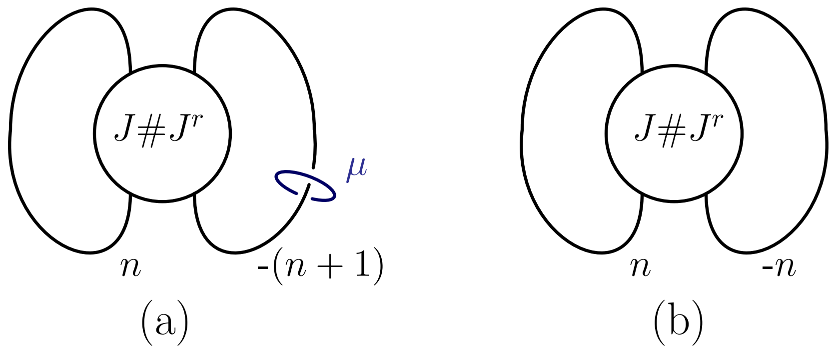


Figure 3.3.7: (a) The manifold Y shown with the rationally null-homologous meridian μ. (b) The manifold Z, obtained by (-1)–surgery on μ.

$-n^2-n$. We are performing (-1)–surgery on a meridian, μ, to one component of the framed link giving Y. The effect of this surgery is to attach a 0–framed disk to every $(-1,1)$–curve on $\partial N(\mu)$. If we select $n+1$ of these curves, we get the torus link $T_{n+1,n+1}$. Since this is an $(n+1)$–component link and each component is a meridian, it is homologous to $(n+1)\cdot\mu = 0$. So this $T_{n+1,n+1}$ bounds an orientable surface in Y. If we attach 0–framed 2-handles to each component, it is easy to see that the intersection among these disks is simply given by the total linking of the components of $T_{n+1,n+1}$. Let S_2 be the surfaces obtained by capping off the $n+1$ boundary components of this orientable surface with these 0–framed disks. Then, $S_2 \cdot S_2 = -n(n+1)$.

The following example will be pertinent to our calculations in Section 3.6.

Example 3.3.6. If J is the unknot, then

$$\begin{aligned} X &= L(n,1), \\ Y &= S^3_{n^2+n}(T_{n,n+1}) = L(n,1)\#L(n+1,-1), \text{ and} \\ Z &= L(n,1)\#L(n,-1). \end{aligned}$$

In general,

$$\begin{aligned} X &= S^3_n(J\#J), \\ Y &= S^3_{n^2+n}(J\#J\#T_{n,n+1}), \text{ and} \\ Z &= S^3_{n,-n}((J\#J)_{(2,0)}). \end{aligned}$$

3.3.4 Enumerating Spin^c structures

This nice homological algebraic set-up gives us natural enumerations of the Spin^c structures on the manifolds in question. Since X is surgery on a knot in S^3, there is an enumeration of $\mathrm{Spin}^c(X)$ by $i \in \mathbb{Z}_n$. Let $\mathfrak{s}_i \in \mathrm{Spin}^c(X)$ for some $i \in \mathbb{Z}_n$.

Let $[\mathfrak{s}_i, \mathfrak{s}_j] \in \mathrm{Spin}^c(Y)$ denote the Spin^c structure on Y that is cobordant to $\mathfrak{s}_i$ via a Spin^c structure $[\mathfrak{s}_i, \mathfrak{t}_m]$ with

$$\langle c_1([\mathfrak{s}_i, \mathfrak{t}_m]), [S_1]\rangle = 2m + n,$$

where $m \in \mathbb{Z}$ is any integer satisfying $m \equiv j \pmod{n+1}$.

Let $[\mathfrak{s}_i, \mathfrak{s}_k] \in \mathrm{Spin}^c(Z)$ denote the Spin^c structure that is cobordant to $[\mathfrak{s}_i, \mathfrak{s}_j]$ via $[\mathfrak{s}_i, \mathfrak{t}_m]$ with

$$\langle c_1([\mathfrak{s}_i, \mathfrak{t}_m]), [S_2]\rangle = 2m + n(n+1),$$

where $m \in \mathbb{Z}$ is any integer satisfying $m \equiv j \pmod{n+1}$ and $m \equiv k \pmod{n}$.

A key feature of this set-up is that we are given affine identifications:

$$\begin{aligned} \mathrm{Spin}^c(X) &\cong \mathbb{Z}_n \\ \mathrm{Spin}^c(Y) &\cong \mathbb{Z}_n \oplus \mathbb{Z}_{n+1} \\ \mathrm{Spin}^c(Z) &\cong \mathbb{Z}_n \oplus \mathbb{Z}_n, \end{aligned}$$

the first and third of which take the unique spin structure to the identity element.

3.3.5 Remarks about surgery coefficients

In what follows, we will use Heegaard Floer theory to study the manifolds described above. In general, when studying the Heegaard Floer homology of surgeries on knots, calculations become much simpler when dealing with large surgery coefficients. For example, Theorem 3.4.6, which we will use extensively, requires that the surgery coefficient be positive and at least $2g - 1$, where g is the genus of the knot that is being surgered. The purpose of this subsection is to show that this criterion is being met in what follows and to examine the knots $\mathcal{K}_{p,k_p}$, which will be used in Section 3.7 to prove Theorem F.

Let $I_{J,p}$ be the knot formed by infecting $T_{2,p}\#T_{2,-p}$ with J along η, as shown in Figure 3.3.3. Consider $J = \#_k D$, which is a knot of genus k. In order to apply Theorem 3.4.6 to the manifold $X = S^3_p(J\#J)$, we must have $p \geq 2g(J\#J) - 1 = 4k - 1$. In order to apply Theorem 3.4.6 to the manifold $Y = S^3_{p^2+p}(J\#J\#T_{p,p+1})$, we must have

$$p^2 + p \geq 2g(J\#J\#T_{p,p+1}) - 1 = 2\left(2k + \frac{p(p-1)}{2}\right) - 1.$$

So, we must have $p \geq \frac{4k-1}{2}$, i.e., $k \leq \frac{2p+1}{4}$. In Section 3.7, it will be necessary for us to consider knots where $k \geq \frac{p+5}{12}$. Let $k_p = \lceil \frac{p+6}{12} \rceil$, and define $\mathcal{K}_{p,k_p} = I_{\#_{k_p}D,p}$. Then, the manifolds associated to $\mathcal{K}_{p,k_p}$ are surgeries of appropriately large coefficient and k_p is large enough to satisfy the conditions in Section 3.7:

$$4k_p - 1 \leq 4\left[\tfrac{p+6}{12} + 1\right] - 1 = \tfrac{p+18}{3} - 1 \leq p$$

$$\tfrac{4k_p-1}{2} \leq \tfrac{4\left[\frac{p+6}{12}+1\right]-1}{2} = \tfrac{p+15}{6} \leq p$$

These inequalities will be satisfied for large p, and for small p it is easy to see that the condition on k_p can be relaxed. It should be noted that there are, in general, many values of k that will suffice for each value of p, we have simply chosen one that will work for all large values of p.

3.3.6 Linking forms and Question 3.1.1

A knot $K \subset S^3$ is called *stably doubly slice* if there exists a doubly slice knot J such that $K\#J$ is doubly slice. Question 3.1.1 can be rephrased to ask whether there exist stably doubly slice knots that are not doubly slice. In this subsection we show that the correction terms could possibly detect the difference between smoothly doubly slice knots and smoothly stably doubly slice knots.

Analogously, we say that a 3–manifold M *stably embeds* smoothly in S^4 if there is a 3–manifold N that embeds smoothly in S^4 such that $M\#N$ embeds smoothly in S^4. It is not known if such an M must itself embed in S^4.

Give a finite abelian group G, a *linking form* on G is a non-degenerate, symmetric, bilinear form $\lambda : G \times G \to \mathbb{Q}/\mathbb{Z}$. For every rational homology 3–sphere M there is a linking form $\lambda : H_1(M) \times H_1(M) \to \mathbb{Q}/\mathbb{Z}$ defined by Poincaré duality.

Now we will consider *linking triples* (G, λ, f), where G is a finite abelian group, λ is a linking form on G, and $f : G \to \mathbb{Q}$ is a function (not necessarily a homomorphism). Such a triple is called *metabolic* if there is a subgroup $G_1 < G$ with $|G_1|^2 = |G|$ such that $\lambda|_{G_1} \equiv 0$ and $f(G_1) = 0$. The triple is called *hyperbolic* if $G = G_1 \oplus G_2$ with $G_1 \cong G_2$ such that $\lambda|_{G_i} \equiv 0$ and $f(G_i) = 0$ for $i = 1, 2$. Note that the set of linking triples has an additive structure given by orthogonal sum.

Lemma 3.3.7. Let (A, μ, f) and (B, ν, g) be linking triples. If (A, μ, f) and $(A \oplus B, \mu \oplus \nu, f \oplus g)$ are both hyperbolic, then (B, ν, g) is metabolic.

Though we will use the hypotheses that (A, μ, f) and $(A \oplus B, \mu \oplus \nu, f \oplus g)$ are hyperbolic, the result hold if these objects are merely metabolic. The following proof is, in essence, due to Kervaire [Ker71].

Proof. Let $A = A_0 \oplus A_1$ and $A \oplus B = L \oplus M$ be the hyperbolic splittings of A and $A \oplus B$. Let $L_i = L \cap (A_i \oplus B)$ and $M_i = M \cap (A_i \oplus B)$ for $i = 0, 1$. Let B_i^L and B_i^M be the projections of L_i and M_i onto B, respectively. From now on, we will restrict our attention to B_0^L.

Let $b, b' \in B_0^L$. Then there exist $a, a' \in A_0$ such that $a \oplus b, a' \oplus b' \in L$. Then,

$$\nu(b, b') = \mu(a, a') + \nu(b, b') = \mu \oplus \nu(a \oplus b, a' \oplus b') = 0,$$

and

$$g(b) = f(a) + g(b) = f \oplus g(a \oplus b) = 0.$$

Thus, the restrictions of ν and g to the B_0^L vanish. Next we show that $|B_0^L|^2 = |B|$. Consider the following two short exact sequences:

$$0 \longrightarrow L_0 \longrightarrow L \xrightarrow{\pi_{A_1}} L_{A_1} \longrightarrow 0,$$

where $\pi_{A_1} : A \oplus B \to A_1$ is projection onto $A_1 < A$, and

$$0 \longrightarrow L \cap (A_0 \oplus 0) \longrightarrow L_0 \xrightarrow{\pi_B} B_0^L \longrightarrow 0,$$

where $\pi_B : A \oplus B$ is projection onto B.

Next, we claim that $|L \cap (A_0 \oplus 0)||A_0||L_{A_1}| \leq |A|$. Assuming this, we see that

$$|B_0^L| = \frac{|L_0|}{|L \cap (A_0 \oplus 0)|} = \frac{|L|}{|L \cap (A_0 \oplus 0)||L_{A_1}|} \geq \frac{|L||A_0|}{|A|} = |B|^{1/2}.$$

Because B_0^L is isotropic and ν is non degenerate, we have that $|B_0^L|^2 = |B|$, as desired. To justify claim assumed above, we will prove that $L \cap (A_0 \oplus 0)$ is orthogonal to $A_0 \oplus L_{A_1}$ under μ. Clearly, $L \cap (A_0 \oplus 0) \perp A_0$. Let $u \in L \cap (A_0 \oplus 0)$ and $w \in L_{A_1}$. Then there exists $v \oplus x \in L_0$ such that $(v + w) \oplus x \in L$. Then,

$$\mu(u, w) = \mu(u, w) + \mu(u, v) = \mu(u, w + v) + \nu(0, x) = \mu \oplus \nu(u \oplus 0, (w + v) \oplus x) = 0.$$

This shows that B_0^L is a metabolizing summand of B. The same is true for B_1^L, B_0^M, and B_1^M. $\square$

Note that the four metabolizers produced in the proof above are all isomorphic. This follows from the classification of linking forms, specifically the fact that a linking form splits over the homogeneous p–group components of the group ([Wal63]). Because of this, we could have performed the above analysis one homogeneous p–group component at a time, each of which would split via L and M.

Now, suppose that A and B are homogeneous p–groups with a common exponent and have ranks $2r$ and $2s$, respectively. Without loss of generality, we can write

$$L = \langle (a_1, b_1), \ldots, (a_t, b_t), (0, b_{t+1}), \ldots, (0, b_{t+l}), (a_{t+l+1}, 0), \ldots, (a_{r+s}, 0),$$

where the b_i are linearly independent, and the a_j are linearly independent. Let $t' = r - l$. Without loss of generality, we can assume that $a_1, \ldots, a_{t'} \in A_0$ and $a_{t'+1}, \ldots, a_{2t'} \in A_1$ (by consideration of the ranks of B_0^L and B_1^L). Since ν is non-degenerate, we can assume that, for $0 \leq i \leq t'$ and $t' + 1 \leq j \leq 2t'$, $\nu(b_i, b_j) \neq 0$ if and only if $j = t' + i$ (perform change of bases within these rank t' summands). Note that $B/\langle b \rangle^\perp$ has rank one for each $b \in B$.

Clearly, $t+l \leq 2s$, and, in fact, we have that t is even with $t/2+l=r$, i.e., $t=2t'$. This claim follows from the ν being non-degenerate; if $t/2 < r-l$, there is an element $(a_{2t'+1}, b_{2t'+1})$ that an be assumed to have the property that $\nu(b_{2t'+1}, b_i) = 0$ for all $0 \leq i \leq t'$. However, if this were the case, then $\langle b_1, \ldots, b_{t'}, b_{2t'+1}, b_{t+1}, \ldots, b_{t+l}\rangle$ would have rank $r+1$ and be isotropic, a contradiction.

It follows that each a_i for $0 \leq i \leq t$ is in either A_0 or A_1. Together with a similar argument for M, we get that $\pi_B(L_0+L_1+M_0+M_1) = B$. In particular, $B = B_0^L + B_1^L + B_0^M + B_1^M$. We can use this to prove a simple corollary.

Corollary 3.3.8. Let (A, μ, f) and (B, ν, g) be linking triples. If (A, μ, f) and $(A \oplus B, \mu \oplus \nu, f \oplus g)$ are both hyperbolic, and if each homogeneous p–group component of B is at most rank 4, then (B, ν, g) is metabolic.

Proof. Since B is rank 4 and spanned by four metabolizers of rank 2 (by the comments above), either some pair of the metabolizers are disjoint, or there is an element b common to each of the four metabolizers. However, the latter case implies that $(0, b) \in L \cap M$, a contradiction. Thus, there is a pair giving a hyperbolic splitting of (B, ν, g). If B is rank 2, a similar argument works. $\square$

Next, we give a counterexample that shows that Corollary 3.3.8 is as strong as possible, in some sense.

Example 3.3.9. Let $A \cong \mathbb{Z}_p^6 = \langle z_1, w_1, z_2, w_2, z_3, w_4\rangle$ and let $B \cong \mathbb{Z}_p^6 = \langle x_1, y_1, x_2, y_2, x_3, y_4\rangle$. Let $A_0 = \langle z_1, z_2, z_3\rangle$ and $A_1 = \langle w_1, w_2, w_3\rangle$. With respect to these bases, let μ and ν be linking forms given by

$$\bigoplus_3 \begin{pmatrix} 0 & -2/p \\ -2/p & 0 \end{pmatrix} \text{ and } \bigoplus_3 \begin{pmatrix} 0 & 2/p \\ 2/p & 0 \end{pmatrix},$$

respectively. Consider the splitting $A \oplus B = L \oplus M$, where

$$L = \langle (z_1, x_1), (z_2, x_2), (w_1, y_1), (w_2, y_2), (0, x_3), (w_3, 0)\rangle,$$

and

$$M = \langle (z_1, y_2), (z_3, x_1), (w_1, x_2), (w_3, y_1), (0, y_3), (w_2, 0)\rangle.$$

It is straightforward to check that $L \cap M = 0$ and that $L + M = A \oplus B$. Furthermore, it is obvious that $\mu \oplus \nu$ vanishes on both L and M. Next, notice that

$$\begin{aligned} B_0^L &= \langle x_1, x_2, x_3\rangle \\ B_1^L &= \langle y_1, y_2, x_3\rangle \\ B_0^M &= \langle x_1, y_2, y_3\rangle \\ B_1^M &= \langle y_1, x_2, y_3\rangle. \end{aligned}$$

No pair of these metabolizers is disjoint. Define $g : B \to \mathbb{Q}$ by

$$g(b) = \begin{cases} 0 & \text{if } b \in B_0^L \cup B_1^L \cup B_0^M \cup B_1^M, \\ 1 & \text{otherwise} \end{cases}.$$

Define $g : B \to \mathbb{Q}$ by

$$f(a) = \begin{cases} -g(b_a) & \text{if } a \notin A_0 \cup A_1, \\ 0 & \text{if } a \in A_0 \cup A_1 \end{cases},$$

Where $a \mapsto b_a$ its the isomorphism from A to B that sends the z_i to the x_i and the w_i to the y_i.

With this set up, it is clear that (A, μ, f) is hyperbolic and that $g : B \to \mathbb{Q}$ is not hyperbolic. It remains to show that $f \oplus g : A \oplus B \to \mathbb{Q}$ vanishes on L and M. This will imply that $(A \oplus B, \mu \oplus \nu, f \oplus g)$ is hyperbolic, thus exemplifying the necessity of the rank restriction in Corollary 3.3.8.

Let $l \in L \cup M$ with $l = (a, b)$. It suffices to check that $f(a) = 0$ if b is in one of the metabolizers listed above and that $a \notin A_0 \cup A_1$ if b is not in one of these metabolizers. It is straightforward to check that these criteria are met.

Let $K \subset S^3$, and let $\mathcal{A} = (A, \mu, f)$ be the linking triple associated to $\Sigma_2(K)$, i.e., $A = H_1(\Sigma_2(K))$, μ is the linking from on A, and $f(a) = d(\Sigma_2(K), \mathfrak{s}_a)$. Let $\mathcal{A}_{p^k}$ denote the restriction of this triple to the homogeneous p^k–group component of A. We have shown the following.

Proposition 3.3.10. Let $K \subset S^3$ and let $\mathcal{A}$ be the associated linking triple. Suppose that $\det(K) = |A| = p_1^{k_1} \cdots p_n^{k_n}$.

1. If K is smoothly doubly slice, then $\mathcal{A}$ is hyperbolic.

2. If K is smoothly stably doubly slice, then $\mathcal{A}_{p_i^{k_i}}$ is hyperbolic whenever $k_i \leq 4$.

Note that (1) is a restatement of Theorem 3.2.2. We will use this result in Sections 3.6 and 3.7 to help prove Theorems E and F.

3.4 Heegaard Floer homology

Below, we collect some basic facts about the suite of invariants known as Heegaard Floer homology. For complete details, see (for example) [OS03a, OS04a, OS04b]. Throughout, let $\mathbb{F}$ denote the field with 2 elements.

3.4.1 3–manifold invariants

Let M be a closed 3–manifold, and let $\mathfrak{s} \in \mathrm{Spin}^c(M)$ be a torsion Spin^c structure on M. Heegaard Floer homology theory associates to $(M, \mathfrak{s})$ a $\mathbb{Z}$–filtered, $\mathbb{Q}$–graded chain complex CF^∞, which is well-defined up to filtered chain homotopy equivalence. This complex is a free, finitely generated $\mathbb{F}[U, U^{-1}]$–module. The action of U lowers the filtration level by one, and lowers the grading by two. Henceforth, if C is any filtered, graded chain complex, then $C_{\{i \leq n\}}$ denotes the subcomplex consisting of elements of filtration level at most n.

Denote the associated homology groups by $HF^\infty(M, \mathfrak{s})$. If M is a rational homology 3–sphere, it turns out that these homology groups are uninteresting. Let $\mathcal{T}^\infty = \mathbb{F}[U, U^{-1}]$. Then, for any rational homology 3–sphere M and any $\mathfrak{s} \in \mathrm{Spin}^c(M)$, we get $HF^\infty(M, \mathfrak{s}) \cong \mathcal{T}^\infty$. This means that any interesting information about $(M, \mathfrak{s})$ must be stored at the chain complex level.

Indeed, there are associated sub- and quotient-complexes:

$$CF^-(M, \mathfrak{s}) = CF^\infty(M, \mathfrak{s})_{\{i<0\}},$$

$$CF^+(M,\mathfrak{s}) = CF^\infty(M,\mathfrak{s})/CF^-(M,\mathfrak{s})$$

and

$$\widehat{CF}(M,\mathfrak{s}) = CF^\infty(M,\mathfrak{s})_{\{i\leq 0\}}/CF^-(M,\mathfrak{s}).$$

The corresponding homology groups, $HF^-(M,\mathfrak{s})$, $HF^+(M,\mathfrak{s})$, and $\widehat{HF}(M,\mathfrak{s})$ turn out to be very powerful 3–manifold invariants. These groups are related by two important long exact sequences:

$$\cdots \longrightarrow HF^-(M,\mathfrak{s}) \xrightarrow{\iota} HF^\infty(M,\mathfrak{s}) \xrightarrow{\pi} HF^+(M,\mathfrak{s}) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow \widehat{HF}(M,\mathfrak{s}) \xrightarrow{\hat{\iota}} HF^+(M,\mathfrak{s}) \xrightarrow{U} HF^+(M,\mathfrak{s}) \longrightarrow \cdots .$$

Note that $\widehat{HF}(M,\mathfrak{s})$ is a finitely generated $\mathbb{F}$-vector space. Define

$$HF_{red}(M,\mathfrak{s}) = HF^+(M,\mathfrak{s})/\mathrm{Im}(\pi).$$

Let $\mathcal{T}^+ = \mathbb{F}[U,U^{-1}]/U\cdot\mathbb{F}[U]$. If M is a rational homology 3–sphere, we have the following decomposition:

$$HF^+(M,\mathfrak{s}) = \mathcal{T}^+ \oplus HF_{red}(M,\mathfrak{s}).$$

It turns out that the grading of the element of lowest grading living in $\mathcal{T}^+$, which we call the tower part of $HF^+(M,\mathfrak{s})$, is an interesting invariant called the *correction term*.

Definition 3.4.1. The *correction term* (or *d–invariant*) of $(M,\mathfrak{s})$ is denoted $d(M,\mathfrak{s})$ and is given by

$$\min\{gr(\pi(\alpha)) \ : \ \alpha \in HF^\infty(M,\mathfrak{s})\}.$$

The correction term enjoys a number of nice properties, including the fact that d is a Spin^c rational homology cobordism invariant (see [OS03a]):

1. $d(M_1 \# M_2, \mathfrak{s}_1 \# \mathfrak{s}_2) = d(M_1, \mathfrak{s}_1) + d(M_2, \mathfrak{s}_2)$,

2. $d(-M, \mathfrak{s}) = -d(M, \mathfrak{s})$, where $-M$ denotes M with the opposite orientation, and

3. $d(M, \mathfrak{s}) = 0$ whenever $(M, \mathfrak{s}) = \partial(W, \mathfrak{t})$, where W is a rational-homology 4–ball, and $\mathfrak{t}|_{\partial W} = \mathfrak{s}$.

This last property is key in the proof of Theorem 3.2.2.

As mentioned above, there are affine identifications

$$\mathrm{Spin}^c(M) \cong H^2(M; \mathbb{Z}),$$

so a rational homology 3–sphere M will have $|H^2(M)|$ correction terms. We will denote the collection of correction terms associated to M by $\mathcal{D}(M)$. When possible, the group structure of $H^2(M)$ will be implicit in our presentation of $\mathcal{D}(M)$. For example, in [OS03a] a formula for the correction terms of lens spaces is given. In particular,

$$d(L(p,1), i) = \frac{p - (2i-p)^2}{4p}. \tag{3.1}$$

Example 3.4.2. Consider the case from Section 3.3 when J is unknotted and $n = 5$. Then, $Y = L(5,1)$, and Equation 3.1 tells us that,

$$\mathcal{D}(L(5,1)) = \{1, 1/5, -1/5, -1/5, 1/5\}.$$

By the additivity of the correction terms, we have the following:

$$\mathcal{D}(L(5,1)\#L(5,-1)) = \left\{ \begin{array}{ccccc} 0 & 4/5 & 6/5 & 6/5 & 4/5 \\ -4/5 & 0 & 2/5 & 2/5 & 0 \\ -6/5 & -2/5 & 0 & 0 & -2/5 \\ -6/5 & -2/5 & 0 & 0 & -2/5 \\ -4/5 & 0 & 2/5 & 2/5 & 0 \end{array} \right\}.$$

Note that implicit in the presentation matrix is the affine identification $\mathrm{Spin}^c(L(n,1)\#L(n,-1)) \cong \mathbb{Z}_5 \oplus \mathbb{Z}_5$ given by $[\mathfrak{s}_i, \mathfrak{s}_j] \sim (i,j)$. For example, the correction terms vanish on all elements of the subgroups generated by $(1,1)$ and $(1,4)$ in $\mathbb{Z}_5 \oplus \mathbb{Z}_5$.

It will sometimes be helpful to write such collections as follows:

$$\mathcal{D}(L(5,1)) = \{-1/5, 1/5, 1, 1/5, -1/5\}.$$

and

$$\mathcal{D}(L(5,1)\#L(5,-1)) = \left\{ \begin{array}{ccccc} 0 & -2/5 & -6/5 & -2/5 & 0 \\ 2/5 & 0 & -4/5 & 0 & 2/5 \\ 6/5 & 4/5 & 0 & 4/5 & 6/5 \\ 2/5 & 0 & -4/5 & 0 & 2/5 \\ 0 & -2/5 & -6/5 & -2/5 & 0 \end{array} \right\}.$$

The only difference here, is that we have centered our indexing set about zero, using

$$\{-(p-1)/2, -(p-3)/2, \ldots, -1, 0, 1, \ldots (p-3)/2, (p-1)/2\}$$

to index $\mathbb{Z}_p$ instead of $\{0, 1, 2, \ldots, p-1\}$.

3.4.2 The surgery exact triangle and 4–dimensional cobordisms

A Spin^c–cobordism between two Spin^c 3–manifolds induces certain maps between the Heegaard Floer homology groups associate to the two manifolds. We now turn our attention to some aspects of these induced maps.

Let M be a rational homology 3–sphere, and let K be a null-homologous knot in M. Let M_0 be the result of N–surgery on K, and let M_1 be the result of $(N+1)$–surgery on K. This is a special case of a broader context in which the triple (M, M_0, M_1) is called a *triad*. For a discussion relevant to this subsection, see [OS06]. Implicit in this set up is a triple of cobordisms obtained by 2-handle addition (cf. Subsection 3.3.3).

$$\begin{tikzcd}
M \arrow[dr, "W_1"'] & & M_1 \arrow[ll, "W_3"'] \\
& M_0 \arrow[ur, "W_2"'] &
\end{tikzcd}$$

Theorem 3.4.3. Let (M, M_0, M_1) be a triad, then there exist exact triangles relating their Heegaard Floer homologies:

$$\begin{tikzcd}
\widehat{HF}(M) \arrow[dr, "\widehat{F}_1"'] & & \widehat{HF}(M_1) \arrow[ll, "\widehat{F}_3"'] \\
& \widehat{HF}(M_0) \arrow[ur, "\widehat{F}_2"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
HF^+(M) \arrow[dr, "F_1^+"'] & & HF^+(M_1) \arrow[ll, "F_3^+"'] \\
& HF^+(M_0) \arrow[ur, "F_2^+"'] &
\end{tikzcd}$$

These maps are induced by the 2–handle cobordisms relating the triad.

Moreover, the grading shifts associated to these induced maps are given by the following formula:

$$gr(F_i^\circ) = gr(F_i^\circ(x)) - gr(x) = \frac{(c_1(\mathfrak{t}))^2 - 2\chi(W_i) - 3\sigma(W_i)}{4}.$$

This set-up can be applied to the 3–manifolds and 4–dimensional cobordisms introduced in Section 3.3. Below, we will use these exact triangles to understand the Heegaard Floer homology of Z (i.e., M_1) via the Heegaard Floer homology of X and Y (i.e., M and M_0), which are more tractable, since they are each realized by surgery on knots in S^3.

In addition to this nice set-up, we have two important theorems about the behavior of these maps on certain types of cobordisms.

Theorem 3.4.4 ([OS03a])**.** Let W be a cobordism between rational homology 3–manifolds obtained by surgery on a knot such that $b_2^+(W) = 0$. Then $F^\infty_{W,\mathfrak{t}}$ is an isomorphism for all $\mathfrak{t} \in \mathrm{Spin}^c(W)$.

The following theorem is implicit in the work of Ozsváth and Szabó [OS03a], and can also be found in [LS04].

Theorem 3.4.5. Let W be a cobordism induced by attaching a 2–handle to a rational homology 3–sphere, and let $\mathfrak{t} \in \mathrm{Spin}^c(W)$. Suppose that W contains a smoothly embedded, closed, orientable surface Σ with $g(\Sigma) > 0$ such that

$$\Sigma \cdot \Sigma \geq 0 \text{ and } |\langle c_1(\mathfrak{t}), [\Sigma]\rangle| + \Sigma \cdot \Sigma > 2g(\Sigma) - 2.$$

Then $\widehat{F}_{W,\mathfrak{t}}$ is zero.

For example, if the 2–handle attachment occurs along a knot with small genus relative to a large positive framing, the induced map $\widehat{F}_{W,\mathfrak{t}}$ will vanish for all $\mathfrak{t} \in \mathrm{Spin}^c(W)$.

3.4.3 Knot complexes

A rationally null-homologous knot K in M induces a second filtration on $CF^\infty(M, \mathfrak{s})$, which thus becomes a $\mathbb{Z} \oplus \mathbb{Z}$–filtered, $\mathbb{Q}$–graded complex, and is denoted $CFK^\infty(M, K, \mathfrak{s})$. The action of U lowers both filtrations by one, and lowers the grading by two. For our purposes, the most important aspect of this complex is that it can be used to determine the Heegaard Floer homology of surgeries on K.

For a positive integer p, let $\mathfrak{s}_m$ denote the element of $\mathrm{Spin}^c(S^3_p(K))$ which is Spin^c cobordant to the unique Spin^c structure on S^3 via an element $\mathfrak{t}_m \in \mathrm{Spin}^c(W)$ (where W is the 2–handle cobordism induced by p–surgery) satisfying

$$\langle c_i(\mathfrak{t}_m, [S]\rangle + p = 2m,$$

where S denotes a Seifert surface for K, capped off with the core of the 2–handle. Then the following theorem is stated as in [HLR12], but is originally proved in [OS04a].

Theorem 3.4.6. Let K be a knot in S^3, and suppose that $g(K) = g$. Let $p \geq 2g - 1$. Then for all m satisfying $|m| \leq \frac{1}{2}(p-1)$, there is a chain homotopy equivalence of graded complexes over $\mathbb{F}[U]$:

$$CF_k^+(S_p^3(K), \mathfrak{s}_m) \simeq CFK_l^\infty(M, K, \mathfrak{s})_{\{\max(i, j-m) \geq 0\}},$$

where

$$k = l + \frac{p - (2m-p)^2}{2p}.$$

Equation 3.1 can be viewed a special case of this (i.e., when K is the unknot). We make extensive use of this theorem in the calculations required by the proof in Section 3.6, which can be found in Section 3.5.

One corollary of this set-up is that the correction terms of manifolds obtained by surgery on knots can be compared to those of lens spaces. We refer the reader to [NW10, NW12] for a nice development. In short, by considering $CFK^\infty(S^3, K)$, one can define two sequences of nonnegative integers V_k, H_k for $k \in \mathbb{Z}$ satisfying

$$V_k = H_{-k}, \qquad V_k \geq V_{k+1} \geq V_k - 1, \qquad V_{g(K)} = 0.$$

It turns out that the correction terms of surgeries on K are determined by these integers.

Theorem 3.4.7. Let K be a knot in S^3. Then,

$$d(S_p^3(K), i) = d(L(p,1), i) - 2\max\{V_i, H_{i-p}\}.$$

3.5 Knot Floer complex calculations

The goal of this section is to perform the correction term calculations required by the proof in Section 3.6. Throughout, we will let $J = \#_k K$ be the connected sum of m copies of K, where K will always be one of three knots: the unknot; the right-handed trefoil, $T_{2,3}$; or the positive, untwisted Whitehead double of the right-handed trefoil, D. Let $X = S^3_n(J\#J)$ and $Y = S^3_{n^2+n}(J\#J\#T_{n,n+1})$; throughout, n will be a positive odd number. The following facts are collected from two theorems of Hedden, Kim, and Livingston [HKL12, Proposition 6.1, Theorem B.1], and are the basis what follows. We will work with coefficients in $\mathbb{F}_2$ throughout.

Theorem 3.5.1 ([HKL12])**.**

1. The chain complex $CFK^\infty(S^3, D)$ is chain homotopy equivalent to $CFK^\infty(S^3, T_{2,3}) \oplus \mathcal{A}$, where $\mathcal{A}$ is an acyclic subcomplex.

2. The chain complex $CFK^\infty(S^3, \#_m T_{2,3}) \simeq CFK^\infty(S^3, T_{2,3})^{\otimes m}$ is chain homotopy equivalent to $CFK^\infty(S^3, T_{2,2m+1}) \oplus \mathcal{A}'$, where $\mathcal{A}'$ is an acyclic subcomplex.

First, we calculate the correction terms for X when $m = 2k$, the case relevant to our discussion. Recall that the affine identification $\mathrm{Spin}^c(X) \cong \mathbb{Z}_n$ gives rise to a natural indexing of $\mathfrak{s}_i \in \mathrm{Spin}^c(X)$, where $|i| \leq (n-1)/2$. This symmetry of this indexing is advantageous, and will be used here. Let $\mathcal{D}(X)$ denote the collection of correction terms associated to X, i.e.,

$$\mathcal{D}(X) = \{d(X, \mathfrak{s}_{\frac{-n+1}{2}}), d(X, \mathfrak{s}_{\frac{-n+3}{2}}), \ldots, d(X, \mathfrak{s}_{-1}), d(X, \mathfrak{s}_0), d(X, \mathfrak{s}_1), \ldots$$

$$\ldots, d(X, \mathfrak{s}_{\frac{n-3}{2}}), d(X, \mathfrak{s}_{\frac{n-1}{2}})\}.$$

Lemma 3.5.2. Let $X = S^3_n(\#_{2k}K)$, where K is $T_{2,3}$ or $Wh^+(T_{2,3}, 0)$. Then, $\mathcal{D}(L(n,1)) - \mathcal{D}(X)$ is given by

$$\{0, \ldots, 0, 2, 2, 4, 4, \ldots, 2k-2, 2k-2, 2k, 2k, 2k, 2k-2, 2k-2, \ldots, 4, 4, 2, 2, 0, \ldots, 0\}.$$

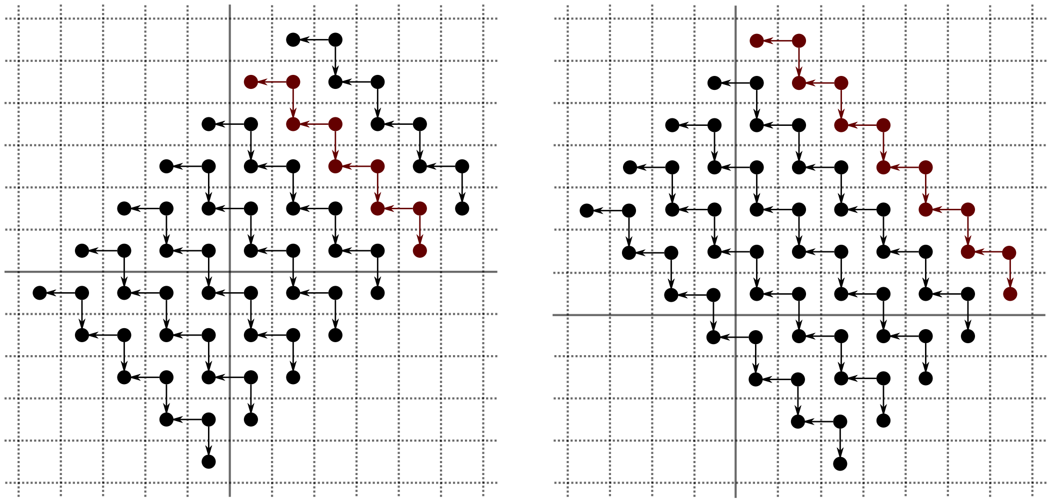
Of course, if K is the unknot, then $\mathcal{D}(X) = \mathcal{D}(L(n,1))$.

Proof. By combining parts (1) and (2) of Theorem 3.5.1, we get that

$$CFK^\infty(S^3, \#_m D) \simeq CFK^\infty(S^3, \#_m T_{2,3}) \oplus \mathcal{A}'' \simeq CFK^\infty(S^3, T_{2,2m+1}) \oplus \mathcal{A}'''.$$

The acyclic pieces can contribute to the homology of $HF^+(X)$, but these contributions are confined to $HF_{red}(X)$ and will not affect the correction term calculations. It follows that $d(X, \mathfrak{s}_i) = d(S^3_n(T_{2,2m+1}), \mathfrak{s}_i)$ for all $|i| \leq (n-1)/2$.

The complex $C = CFK^\infty(S^3, T_{2,2m+1})$ can be easily obtained from the Alexander polynomial $\Delta_{T_{2,2m+1}}(t)$, since $T_{2,2m+1}$ is an alternating L-space knot [OS03b], and is shown in Figure 3.5.1. Its basic building block (which we call a *germ*) can be seen in Figure 3.5.1. One way to characterize which piece of the total complex is the germ G is to say that G is contained in the first (i,j)–quadrant, but UG is not. The total complex is obtained by taking $\mathbb{Z}$ copies of G, which are related by U–translation, i.e., $C = \sqcup_{z\in\mathbb{Z}} U^z G$.



(a)

(b)

Figure 3.5.1: Portions of the complex $CFK^\infty(S^3, T_{2,2m+1})$ is shown above for (a) $m = 4$ and (b) $m = 6$. The germ of each complex is shown in red.

There is a simple way to calculate $V_l = V_l(T_{2,2m+1})$ in this case. Consider the subcomplex $C_{\{\max(i,j-l)\geq 0\}}$. Then,

$$V_l = \max\{z \ : \ U^z G \cap C_{\{\max(i,j-l)\geq 0\}} = \emptyset\}.$$

See, for example, Figure 3.5.2. With this in mind, it is now easy to see that

$$\{V_l(T_{2,2m+1})\}_{l\geq 0} = \begin{cases} \{k, k, k-1, k-1, \ldots, 2, 2, 1, 1, 0, \ldots\} & \text{if } m = 2k, \\ \{k, k-1, k-1, \ldots, 2, 2, 1, 1, 0, \ldots\} & \text{if } m = 2k+1, \end{cases}$$

where each value less than k appears twice in each list, and the infinite tails each consists of zeros. If we recall that $H_l = V_{-l}$, then Theorem 3.4.7 completes the proof. $\square$

Let L_k denote the list of even integers given in Lemma 3.5.2, but with each value halved, and consider the bijection between L_k and $\mathbb{Z}$ where the

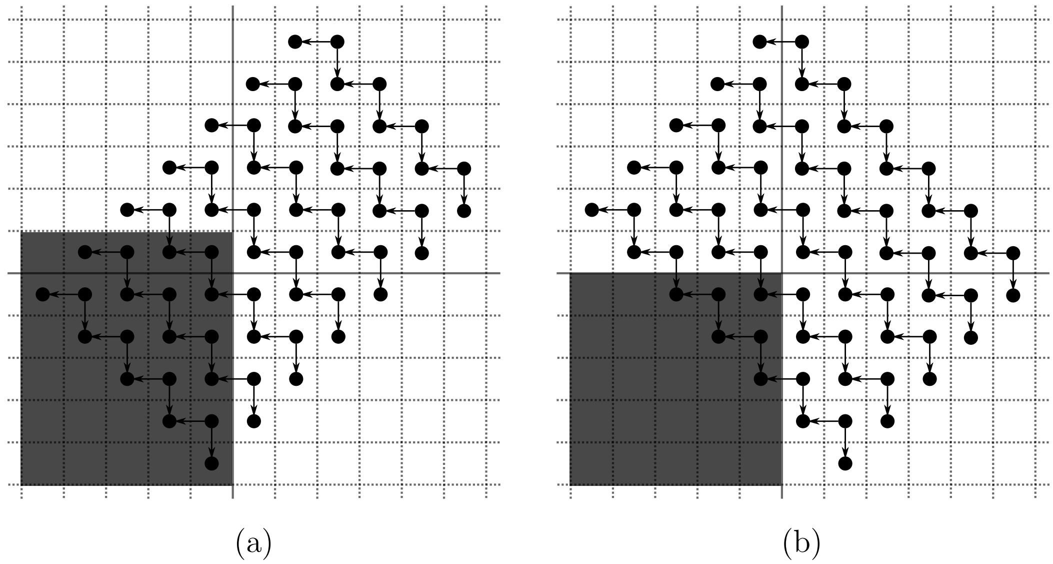


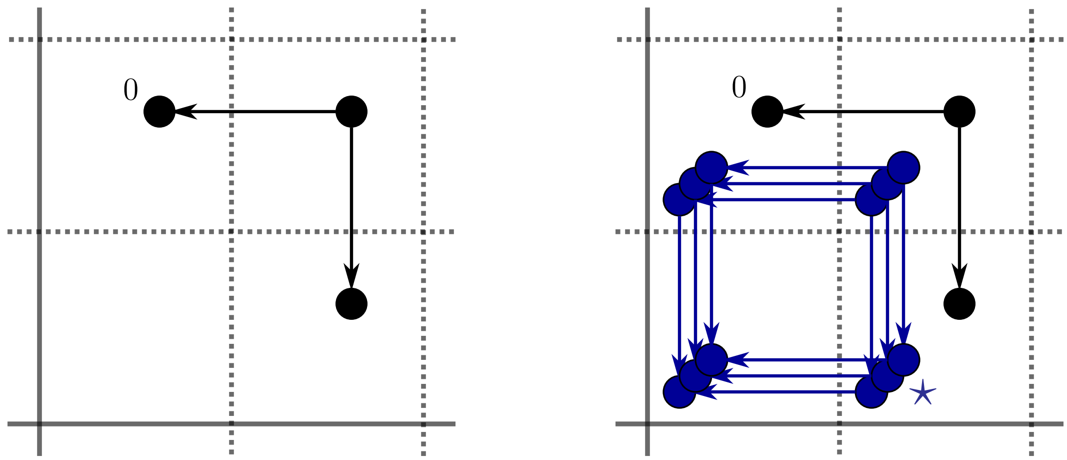
Figure 3.5.2: (a) The calculation showing that $V_1(T_{2,9}) = 2$. (b) The calculation showing that $V_0(T_{2,13}) = 3$.

central k corresponds to zero and the values to the left and right correspond to the negative and positive integers, respectively. Let L_k^t denote a truncated version of L_k where the value of any term in L_k^t corresponding to an integer less than $-t$ is set to zero. Let $L_k^t(x)$ represent the element of L_k^t corresponding to $x \in \mathbb{Z}$. For example,

$$\begin{aligned} L_3 &= \{\ldots, 0, 0, 1, 1, 2, 2, 3, 3, 3, 2, 2, 1, 1, 0, 0 \ldots\}, \\ L_3^1 &= \{\ldots, 0, 0, 0, 0, 0, 0, 3, 3, 3, 2, 2, 1, 1, 0, 0 \ldots\}, \\ L_3^3 &= \{\ldots, 0, 0, 0, 0, 2, 2, 3, 3, 3, 2, 2, 1, 1, 0, 0 \ldots\}, \end{aligned}$$

and $L_3^1(-1) = 3$. We will make use of these truncated lists later.

Our next task is to give a calculation for the correction terms of Y. To start, consider the case when K is the unknot, so $Y = S^3_{n^2+n}(T_{n,n+1})$.



(a) (b)

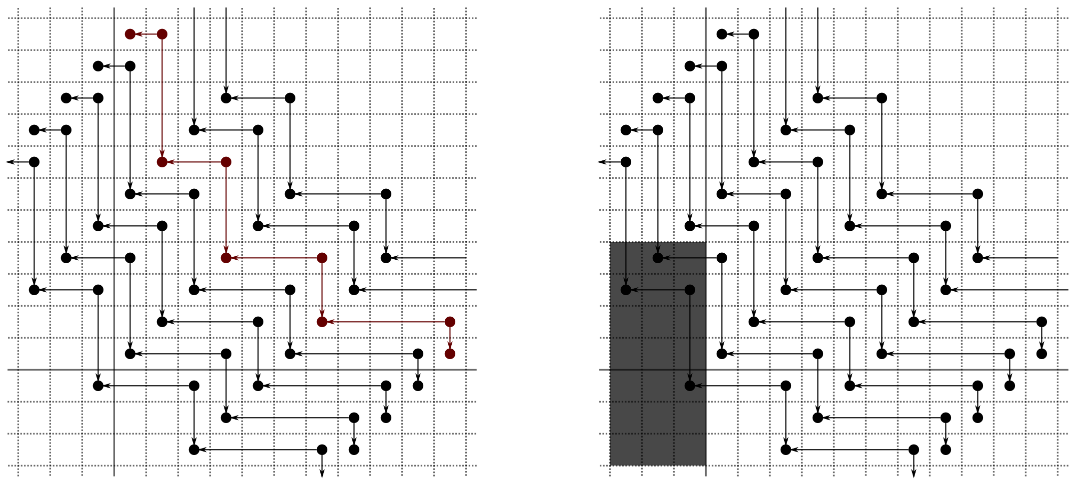
Figure 3.5.3: The complexes (a) $CFK^\infty(S^3, T_{2,3})$ and (b) $CFK^\infty(S^3, D)$ are shown with gradings; the three chains adjacent to the star have gradings $-2, -2$, and -1.

Lemma 3.5.3. Let $n = 2d+1$ for $d \in \mathbb{N}$. Then, $\{V_l(T_{n,n+1})\}_{l\geq 0}$ is given by

$$
\begin{array}{lllllllll}
\{ & Tr(d), & Tr(d), & \dots & Tr(d), & Tr(d)\text{-}1, & Tr(d)\text{-}2, & \dots, & Tr(d\text{-}1)\text{+}2, \quad Tr(d\text{-}1)\text{+}1, \\
 & Tr(d\text{-}1), & Tr(d\text{-}1), & \dots & Tr(d\text{-}1), & Tr(d\text{-}1)\text{-}1, & Tr(d\text{-}1)\text{-}2, & \dots, & Tr(d\text{-}2)\text{+}2, \quad Tr(d\text{-}2)\text{+}1, \\
 & Tr(d\text{-}2), & Tr(d\text{-}2), & \dots & Tr(d\text{-}2), & Tr(d\text{-}2)\text{-}1, & Tr(d\text{-}2)\text{-}2, & \dots, & Tr(d\text{-}3)\text{+}2, \quad Tr(d\text{-}3)\text{+}1, \\
 & & & & & \vdots & & & \\
 & 3, & 3, & 3, & \dots, & & \dots, & 3, & 3, \quad 2, \\
 & 1, & 1, & 1, & \dots, & & \dots, & 1, & 1, \quad 1, \\
 & 0, & 0, & 0, & \dots, & \}, & & &
\end{array}
$$

where $Tr(k)$ denotes the k^{th} triangular number.

To clarify, the above list has been displayed so as to make the pattern of its elements more clear. On the i^{th} line, the value $Tr(d-i+1)$ appears $d+i+1$ times, followed by sequential decreases by 1, until the next triangular number is hit, which begins a new line. The tail of the list is all zeros. We will refer to the first appearance of each triangular number (i.e., the first element of each line) as a *pivot*. These pivots occur when l is a multiple of n and correspond



(a) (b)

Figure 3.5.4: (a) The complex $CFK^\infty(S^3, T_{n,n+1})$; here $n = 5$. The calculation showing that $V_4(T_{5,6}) = 2$.

to the cycles in the germ G of $CFK^\infty(S^3, T_{n,n+1})$ (see Figure 3.5.4).

Proof. As noted in [HKL12], $C = CFK^\infty(S^3, T_{n,n+1})$ has germ G as shown in red in Figure 3.5.4(a). The total complex is obtained by taking $\mathbb{Z}$ copies of this germ, which are related by U–translation, i.e., $C = \cup_{z\in\mathbb{Z}} U^z G$. As in the proof of Lemma 3.5.2, the $V_l = V_l(T_{n,n+1})$ are given by

$$V_l = \max\{z \ : \ U^z G \cap C_{\{\max(i,j-l)\geq 0\}} = \emptyset\}.$$

See for example, Figure 3.5.4(b). Putting all this together, it is easy to see that $\{V_l\}_{l\geq 0}$ is as claimed. $\square$

Lemma 3.5.3 gives us a basis to understand the correction terms for surgeries on $J\#J\#T_{n,n+1}$. To continue, we need to understand how the knot chain complex for $T_{n,n+1}$ changes under connected sum with K.

Lemma 3.5.4. After a filtration-preserving change of basis,

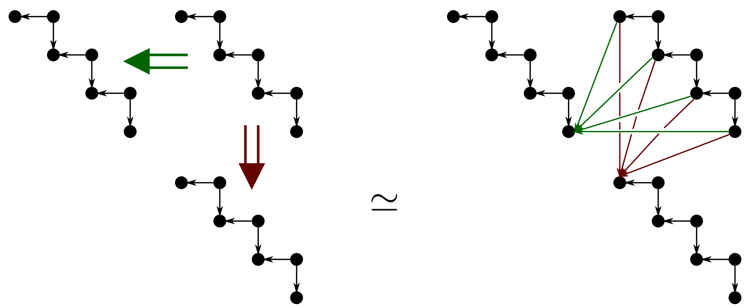
$$CFK^\infty(S^3, J\#J\#T_{n,n+1}) = C_{sum} \oplus \mathcal{A},$$

where a germ for C_{sum} is made up of the the characteristic pieces shown in Figure 3.5.6, and $\mathcal{A}$ is an acyclic subcomplex.

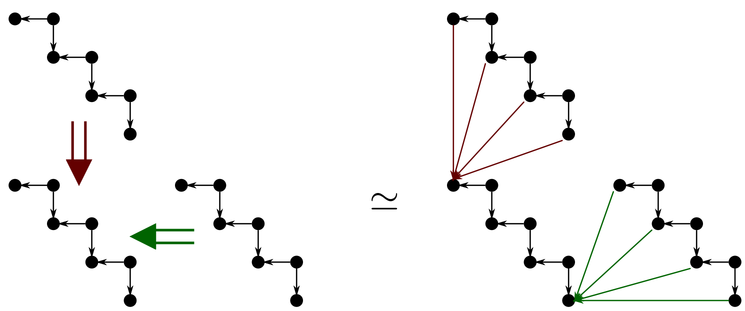
Proof. Recall that $CFK^\infty(S^3, J\#J) \simeq CFK^\infty(S^3, T_{2,2m+1}) \oplus \mathcal{A}$, by Theorem 3.5.1. It follows that $CFK^\infty(S^3, J\#J\#T_{n,n+1}) \simeq CFK^\infty(S^3, T_{n,n+1}) \otimes CFK^\infty(S^3, T_{2,2m+1}) \oplus \mathcal{A}$. (Here use $\mathcal{A}$ to represent potentially different acyclic subcomplexes.) To see that this tensor product has the desired form, we need three pictorial lemmas, shown in Figure 3.5.5. All three parts show a chain homotopy equivalence achieved via a filtration preserving change of basis. Consider Figure 3.5.5(a), and denote the chains by $a_1, \ldots, a_{2m+1}$, $b_1, \ldots, b_{2m+1}$, and $c_1, \ldots, c_{2m+1}$ (i.e., $b_i \mapsto a_i + c_i$). The double arrows mean that there should be an arrow between each pair of vertically or horizontally aligned chains. The pertinent change of basis is:

$$b_k \mapsto b_k + \sum_{\substack{i<k,\\ i \text{ even},\\ n_j(b_k)>n_j(c_i)}} c_i + \sum_{\substack{j>k,\\ j \text{ even}\\ n_i(b_k)>n_i(a_i)}} a_j,$$

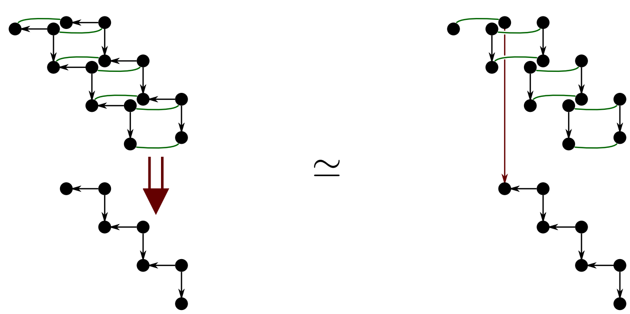
for odd k. (Note that the indexing variables i, j, and k used here are not related to the uses of i, j, and k used elsewhere; in particular, i and n_i are not related here.) The third condition on each summation guarantees that this change of basis is filtration preserving. Note that any vertical arrows hitting the a–group or horizontal arrows hitting the c–group are unaffected,



(a)



(b)



(c)

Figure 3.5.5: The three filtered chain homotopy equivalences used in the proof of Lemma 3.5.4. Each is obtained by a filtration-preserving change of basis.

since the chains in these groups are unchanged. The result is that the only arrows from the b–group to either the a–group or the c–group will go from b_k with odd k to a_i and c_j with odd i and odd j. Parts (a) and (b) of Figure 3.5.6 show the possible results of this local change of basis on the germ of $CFK^\infty(S^3, J\#J\#_{n,n+1})$. In (a), the a and b pieces overlap; in (b) they do not.

Next, Figure 3.5.5(b) corresponds to the filtration preserving change of basis given by

$$c_k \mapsto c_k + \sum_{\substack{j>k,\\ i \text{ even},\\ n_j(b_j)>n_j(c_k)}} b_j,$$

for odd k, and

$$a_k \mapsto a_k + \sum_{\substack{i<k,\\ i \text{ even}\\ n_i(b_i)>n_i(a_k)}} b_i,$$

for odd k.

Finally, Figure 3.5.5(c) corresponds to the filtration preserving change of basis given by

$$a_k \mapsto a_k + c_1,$$

for odd $k \geq 3$,

$$a_k \mapsto a_k + b_{k-1} + \sum_{\substack{i<k,\\ i \text{ even}}} c_i,$$

for even k, and

$$b_k \mapsto b_k + \sum_{\substack{i<k,\\ i \text{ even}}} c_i,$$

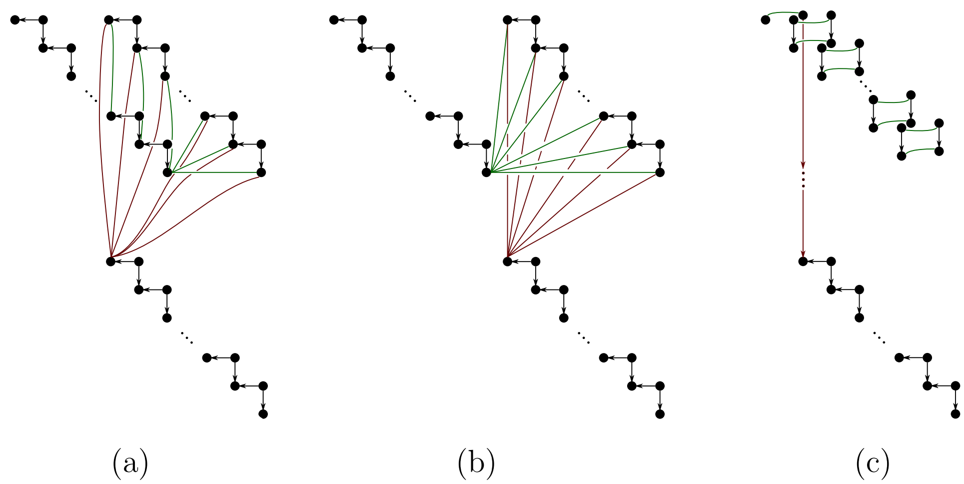


Figure 3.5.6: Three characteristic pieces of the complex $C_{T_{n,n+1}}$.

for odd $k > 2$. See also Figure 3.5.6(c).

By applying these three types of change of basis, we see that $CFK^\infty(S^3, J\#J\#T_{n,n+1}) = C_{sum} \oplus \mathcal{A}$, where the characteristic pieces of C_{sum} are shown in Figure 3.5.6. In other words, connected summing $T_{n,n+1}$ with $J\#J$ introduces a stepping pattern at every joint (i.e., the cycles) of the germ of $CFK^\infty(S^3, T_{n,n+1})$, except the first and last, where the germ simply extends by m. $\square$

Lemma 3.5.5. Let K be $T_{2,3}$ or D and let $J = \#_k K$. Then, for $|l| \leq \frac{n^2+n}{2}$,

$$V_l(J\#J\#T_{n,n+1}) - V_l(T_{n,n+1}) = \begin{cases} L_k^{t(l)}(\bar{l}) & \text{if } \ 0 \leq l \leq \frac{n(n-1)}{2}, \\ 1 & \text{if } \ \frac{n(n-1)}{2} \leq l \leq \frac{n(n-1)}{2} + k, \\ 0 & \text{if } \ \frac{n(n-1)}{2} + k < l, \end{cases}$$

where $\bar{l} \in [\frac{-n+1}{2}, \frac{n-1}{2}]$ is the mod n reduction of l, and $t(l) = \frac{n-3}{2} - a$ if $|l| \in [an - \frac{n-1}{2}, an + \frac{n+1}{2})$.

Proof. Lemma 3.5.4 tells us that the germ of $CFK^\infty(S^3, J\#J\#T_{n,n+1})$ is given locally as in Figure 3.5.6. Forming the connected sum changes the complex for $T_{n,n+1}$ by introducing a stepping pattern at each joint. It is straightforward to see the effect of this on the $V_l(T_{n,n+1})$. For each joint, one simply superimposes a copy of L_k^t over $\{V_l(T_{n,n+1})\}_{\geq 0}$, with $L_k(0)$ centered over the l corresponding to the joint. If C_{sum} looks locally like Figure 3.5.6(a) at the joint (i.e., there is some overlap), then we use L_k^t where $t-1$ is the amount of overlap (in Figure 3.5.6(a), the overlap shown is 3). If C_{sum} looks locally like Figure 3.5.6(b) at the joint (i.e., there is no overlap), then $L_k^t = L_k$ (i.e., there is no truncation).

To clarify, $\{V_l(T_{n,n+1})\}_{\geq 0}$ is shown below, with the pivots highlighted in red. These pivots correspond to the joints of $CFK^\infty(S^3, T_{n,n+1})$. In C_{sum}, each such joint has been tensored with the germ for $CFK^\infty(S^3, T_{2,2m+1})$ and looks locally as in Figure 3.5.6. By considering these local pictures, we can see that $V_l = V_l(J\#J\#T_{n,n_1})$ will have the value claimed, because the introduction of the stepping pattern corresponds precisely to adding $L_k^{t(l)}(\bar{l})$ to V_l.

$$\begin{array}{cccccccccc}
\{ & \color{red}{Tr(d)}, & Tr(d), & \dots & Tr(d), & Tr(d)\text{-}1, & Tr(d)\text{-}2, & \dots, & Tr(d\text{-}1)+2, & Tr(d\text{-}1)+1, \\
 & \color{red}{Tr(d\text{-}1)}, & Tr(d\text{-}1), & \dots & Tr(d\text{-}1), & Tr(d\text{-}1)\text{-}1, & Tr(d\text{-}1)\text{-}2, & \dots, & Tr(d\text{-}2)+2, & Tr(d\text{-}2)+1, \\
 & \color{red}{Tr(d\text{-}2)}, & Tr(d\text{-}2), & \dots & Tr(d\text{-}2), & Tr(d\text{-}2)\text{-}1, & Tr(d\text{-}2)\text{-}2, & \dots, & Tr(d\text{-}3)+2, & Tr(d\text{-}3)+1, \\
 & & & & & \vdots & & & & \\
 & \color{red}{3}, & 3, & 3, & \dots, & & \dots, & 3, & 3, & 2, \\
 & \color{red}{1}, & 1, & 1, & \dots, & & \dots, & 1, & 1, & 1, \\
 & \color{red}{0}, & 0, & 0, & \dots, & \}, & & & &
\end{array}$$

$\square$

Now that we have calculated $\{V_l(J\#J\#T_{n,n+1})\}_{l\geq 0}$, it is straightforward to calculate the correction terms for Y.

Corollary 3.5.6. Let K be $T_{2,3}$ or D, let $J = \#_k K$, and let $Y = S^3_{n^2+n}(J\#J\#T_{n,n+1})$. Then,

$$\mathcal{D}(L(n,1)\#L(n+1,-1)) - \mathcal{D}(Y)$$

is given by

$$\begin{bmatrix}
0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
2 & \cdots & 2 & {\color{red}2} & 0 & {\color{red}0} & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & \cdots & 2 & 2 & 2 & \cdots & 2 \\
2 & \cdots & 2 & 2 & {\color{red}2} & 0 & {\color{red}0} & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & \cdots & 2 & 2 & 2 & \cdots & 2 \\
4 & \cdots & 4 & 4 & 4 & {\color{red}2} & 0 & {\color{red}0} & 0 & 0 & 0 & 4 & 4 & 4 & 4 & \cdots & 4 & 4 & 4 & \cdots & 4 \\
4 & \cdots & 4 & 4 & 4 & 4 & {\color{red}2} & 0 & {\color{red}0} & 0 & 0 & 4 & 4 & 4 & 4 & \cdots & 4 & 4 & 4 & \cdots & 4 \\
\vdots & & \vdots & \vdots & \vdots & & & {\color{red}\ddots} & \ddots & {\color{red}\ddots} & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
2k & \cdots & 2k & 2k & 2k & \cdots & 2k & 2k & {\color{red}2} & 0 & {\color{red}0} & 2k & 2k & 2k & 2k & \cdots & 2k & 2k & 2k & \cdots & 2k \\
2k & \cdots & 2k & 2k & 2k & \cdots & 2k & 2k & 2k & {\color{red}2} & 0 & {\color{red}2} & 2k & 2k & 2k & \cdots & 2k & 2k & 2k & \cdots & 2k \\
2k & \cdots & 2k & 2k & 2k & \cdots & 2k & 2k & 2k & 2k & {\color{red}0} & 0 & {\color{red}2} & 2k & 2k & \cdots & 2k & 2k & 2k & \cdots & 2k \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & {\color{red}\ddots} & \ddots & {\color{red}\ddots} & & & \vdots & \vdots & \vdots & & \vdots \\
4 & \cdots & 4 & 4 & 4 & \cdots & 4 & 4 & 4 & 4 & 0 & 0 & {\color{red}0} & 0 & {\color{red}2} & 4 & 4 & 4 & 4 & \cdots & 4 \\
4 & \cdots & 4 & 4 & 4 & \cdots & 4 & 4 & 4 & 4 & 0 & 0 & 0 & {\color{red}0} & 0 & {\color{red}2} & 4 & 4 & 4 & \cdots & 4 \\
2 & \cdots & 2 & 2 & 2 & \cdots & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & {\color{red}0} & 0 & {\color{red}2} & 2 & 2 & \cdots & 2 \\
2 & \cdots & 2 & 2 & 2 & \cdots & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & {\color{red}0} & 0 & {\color{red}2} & 2 & \cdots & 2 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0
\end{bmatrix},$$

where the rows are indexed by $i \in [\frac{-n+1}{2}, \frac{n-1}{2}]$ and the columns are indexed by $j \in [\frac{-n}{2}, \frac{n-2}{2}]$.

This matrix, $\mathcal{M}$ is not as complicated as it looks. It is a $(n \times (n+1))$–matrix, with zeros along the right off-diagonal. If the first column is removed, it is rotationally symmetric. Think of $\mathcal{M}$ as the union of its diagonals. Every diagonal (other than the middle three) is simply a copy of (twice) L^t_k, where t is the displacement (left or right) from the center three diagonals. For example, if we consider the diagonal directly to the left of the middle three diagonals, we see a copy of (twice) L^1_k that emanates towards the northwest.

Proof. The matrix presentation for $\mathcal{D}(L(n,1)\#L(n+1,-1)) - \mathcal{D}(Y)$ follows from Lemma 3.5.5 in light of the following remark. Knowing $V_l(J\#J\#T_{n,n+1})$ allows us to calculate $d(Y,\mathfrak{s}_l)$, but in the matrix above, we have given $d(Y,[\mathfrak{s}_i,\mathfrak{s}_j])$, which uses the enumeration of Spin^c structures introduced in Subsection 3.3.4. This correspondence is given by $l = \frac{n(n+1)}{2} + (n+1)i - nj$, where l, i, and j are all taken to be centered about zero. This identification maps the subgroup generated by $n+1 \in \mathbb{Z}_{n^2+n}$ to the subgroup generated by $(1,1) \in \mathbb{Z}_n \oplus \mathbb{Z}_{n+1}$. $\square$

There is a slight issue related to our choice of identification. In fact, there are four different identifications we could have chosen, each of which is related to the others by negating i, j, or both. The following lemma proves that these are the only four identifications that we should concern ourselves with, since any other identification will not preserve the equivalence class of the correction terms modulo 2.

Even if we are content with only these four identifications, we need to note that a different choice of identification changes our labeling of the correction terms. We have introduced an indeterminacy in which we cannot distinguish i from $-i$ or j from $-j$ in our labelings. Fortunately, this does not affect the proofs of Theorem E and F.

Lemma 3.5.7. Let $l = \frac{n(n+1)}{2} + (n+1)i - nj$. Then,

$$d(S^3_{n^2+n}(J\#J\#T_{n,n+1}), l) \equiv d(L(n,1), i) - d(L(n+1,1), j) \pmod 2.$$

Proof. By the integer surgery formula, we see that

$$d(S^3_{n^2+n}(T_{n,n+1}), l) \equiv d(L(n^2+n,1), l) \pmod 2.$$

Let $k = j - i$, then it is straightforward to show that

$$\frac{(2l - n(n+1))^2 - n(n+1)}{4n(n+1)} - \frac{(2i-n)^2 - n}{4n} - \frac{(2j-(n+1))^2 - (n+1)}{4(n+1)} = k^2 - k.$$

From this, it follows that $d(L(n^2+n,1),l) \equiv d(L(n,1),i) - d(L(n+1,1),j) \pmod 2$ and that

$$d(S^3_{n^2+n}(T_{n,n+1}), l) \equiv d(L(n,1),i) - d(L(n+1,1),j) \pmod 2.$$

Furthermore, it is easy to see that $d(L(p,1),a) \equiv d(L(p,1),b) \pmod 2$ if and only if $b = p - a$, assuming $0 \leq a < b < p$. Thus, for any Y obtained by surgery on a knot in S^3, we know that $d(Y,a) = d(Y,b)$ if and only if $b = a$ or $p - a$. In other words, the equality of two correction terms is determined by their mod 2 equivalence class for such 3–manifolds. This implies that

$$d(S^3_{n^2+n}(T_{n,n+1}), l) = d(L(n,1),i) - d(L(n+1,1),j).$$

To complete the proof, we simply note that $V_l(J\#J\#T_{n,n+1}) \equiv V_l(T_{n,n+1}) \pmod 2$, by Lemma 3.5.5. $\square$

Descriptions of the germs for the total complexes $CFK^\infty(S^3, \#_m T_{2,3})$ and $CFK^\infty(S^3, \#_m D)$ are given in Figure 3.5.8. These presentation follow

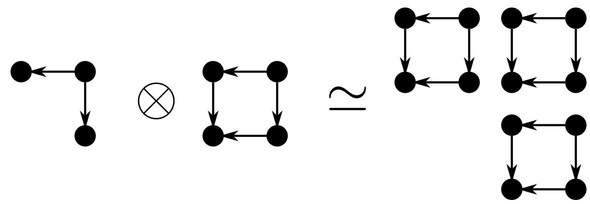


Figure 3.5.7: The filtered chain homotopy equivalence shown above is a straightforward exercise.

from Theorem 3.5.1, the pictorial lemma shown in Figure 3.5.7, and induction. The proof of this pictorial lemma is straightforward. Note that in Figure 3.5.8 each acyclic square shown is meant to represent a multitude of overlying acyclic squares, but the gradings are controlled.

Here is how the gradings behave in Figure 3.5.8. In (a), the gradings are as shown, and overlaid squares have the same gradings as the representative shown. In (b), for each collection of overlaid squares, the maximally graded representative is shown, and there are $m+1$ different possible gradings. For example, consider the bottom-left square in the right part of (b). This square represents many squares, each of which has as its bottom-left corner a chain with grading in $\{-m, -(m+1), -(m+2), \ldots, -2m\}$.

We are now prepared to prove one final property about $HF^+(Y')$, which we will need in order to complete the proof in Section 3.6.

Lemma 3.5.8. Let $Y = S^3_{n^2+n}(J\#J\#T_{n,n+1})$, and let $\xi \in HF_{red}(Y, [\mathfrak{s}_i, \mathfrak{s}_j])$. Then

$$gr(\xi) \leq gr\left(\mathcal{T}^+_{i,j}(Y)\right).$$

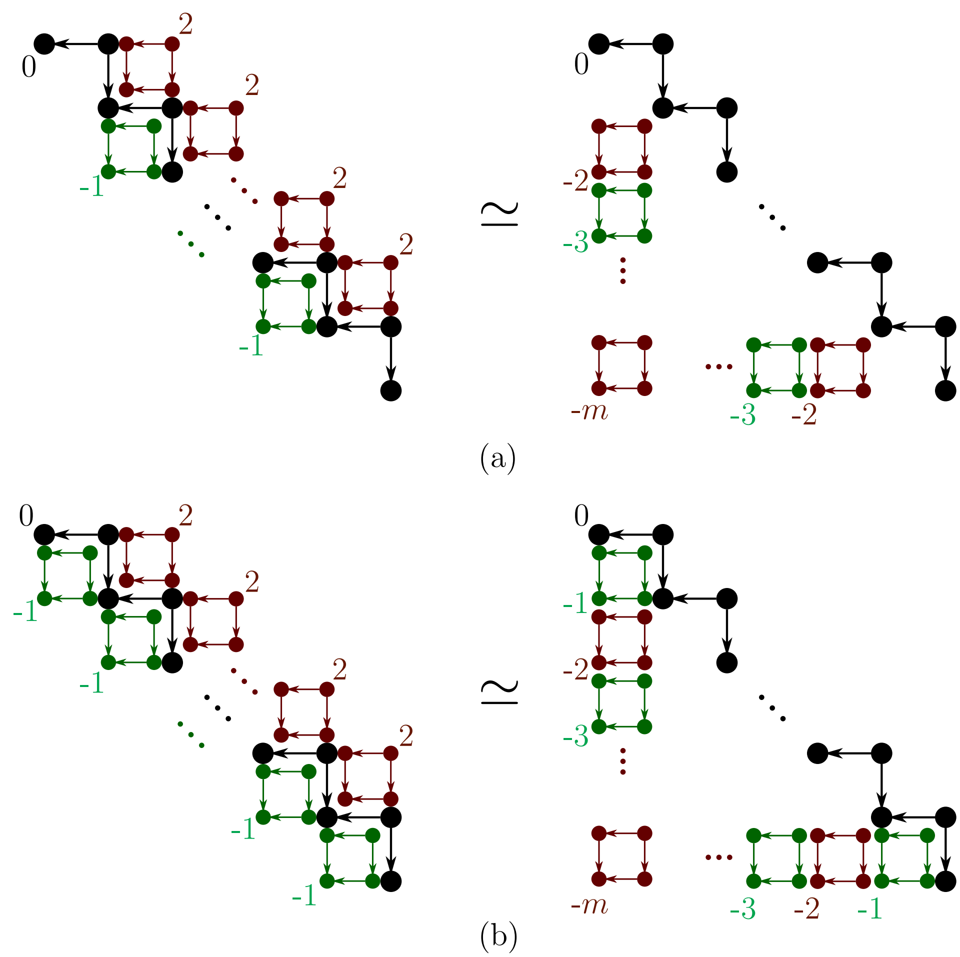


Figure 3.5.8: Germs for the total complexes of (a) $CFK^{\infty}(S^3, \#_m T_{2,3})$ and (b) $CFK^{\infty}(S^3, \#_m D)$. Note that each square above is meant to represent a multitude of overlaid squares. In (a), gradings of overlaid squares match up and are as shown. In (b), gradings in overlaid squares may be lower than shown. (See text.)

Proof. Let $C = CFK^\infty(S^3, J\#J\#T_{n,n+1})$, let $C^1 = CFK^\infty(S^3, J\#J)$, and let $C^2 = CFK^\infty(S^3, T_{n,n+1})$, so $C = C^1 \otimes C^2$. Let G, G^1, and G^2 be the germs for C, C^1, and C^2, respectively. Let $\xi \in HF_{red}(Y, [\mathfrak{s}_i, \mathfrak{s}_j])$, and let $c \in C$ be any chain such that $[c] = \xi$.

Let $c' \in C$ be any chain such that $[c']$ is the element of lowest grading in $\mathcal{T}_{i,j}(Y)$, so $gr(c') = gr(\mathcal{T}_{i,j}(Y))$. Let $G' = U^{z'}G$ be the germ containing c', where $z' \in \mathbb{Z}$. Any chain in $\cup_{e>0} U^e G'$ that is not homologous to a U–translate of c' is not a cycle. To see this, simply observe that $H_*(\cup_{e>0} U^e G') \cong \mathcal{T}^+$, and is generated by U–translates of $[c']$.

Suppose that $c \in U^z G$ for some $z \in \mathbb{Z}$. Since c is a cycle and not homologous to a U–translate of c', we see that $z \geq z'$. Let $c'' = U^{z'-z}c$, so $c'' \in G'$, and let $c'' = c^1 \otimes c^2$ with $c^1 \in G^1$ and $c^2 \in U^{z'}G^2$. Note that

$$0 = \partial c'' = \partial(c^1 \otimes c^2) = \partial c^1 \otimes c^2 + c^1 \otimes \partial c^2.$$

It follows that c^1 and c^2 are both cycles

By considering Figure 3.5.8, we see that any cycle in G^1 has nonpositive grading. Furthermore, any cycle in $U^{z'}G^2$ has grading $-2z'$. Let $c' = c^3 \otimes c^4$, where $c^3 \in G^1$ and $c^4 \in U^{z'}G^2$. Since $[c']$ is the element of lowest grading in $\mathcal{T}_{i,j}(Y)$ it follows that $gr(c^3) = 0$ and $gr(c^4) = -2z'$.

It follows that $gr(c) \leq gr(c'') \leq gr(c')$, as desired.

$\square$

3.6 Proof of Theorem E

In this section, we prove the following proposition, whose corollary, together with Corollary 3.3.5, implies Theorem E. Recall the geometric set-up from Section 3.3. In particular, let $\mathcal{Z}_{p,k_p}$ be the double branched cover of the knot $\mathcal{K}_{p,k_p}$.

Proposition 3.6.1. The difference $\mathcal{D}(L(p,1)\#L(p,-1)) - \mathcal{D}(\mathcal{Z}_{p,k_p})$ is given by the following matrix.

$$\left[\begin{array}{ccccccccccccccccccc}
0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
2 & \cdots & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & \cdots & 2 & 2 & \cdots & 2 \\
2 & \cdots & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & \cdots & 2 & 2 & \cdots & 2 \\
4 & \cdots & 4 & 4 & 2 & 0 & 0 & 0 & 0 & 0 & 4 & 4 & 4 & 4 & \cdots & 4 & 4 & \cdots & 4 \\
4 & \cdots & 4 & 4 & 4 & 2 & 0 & 0 & 0 & 0 & 4 & 4 & 4 & 4 & \cdots & 4 & 4 & \cdots & 4 \\
\vdots & & \vdots & \vdots & & & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
2k_p & \cdots & 2k_p & 2k_p & \cdots & 2k_p & 2k_p & 2 & 0 & 0 & 2k_p & 2k_p & 2k_p & 2k_p & \cdots & 2k_p & 2k_p & \cdots & 2k_p \\
2k_p & \cdots & 2k_p & 2k_p & \cdots & 2k_p & 2k_p & 2k_p & 2 & 0 & 2 & 2k_p & 2k_p & 2k_p & \cdots & 2k_p & 2k_p & \cdots & 2k_p \\
2k_p & \cdots & 2k_p & 2k_p & \cdots & 2k_p & 2k_p & 2k_p & 2k_p & 0 & 0 & 2 & 2k_p & 2k_p & \cdots & 2k_p & 2k_p & \cdots & 2k_p \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \ddots & \ddots & \ddots & & & \vdots & \vdots & & \vdots \\
4 & \cdots & 4 & 4 & \cdots & 4 & 4 & 4 & 4 & 0 & 0 & 0 & 0 & 2 & 4 & 4 & 4 & \cdots & 4 \\
4 & \cdots & 4 & 4 & \cdots & 4 & 4 & 4 & 4 & 0 & 0 & 0 & 0 & 0 & 2 & 4 & 4 & \cdots & 4 \\
2 & \cdots & 2 & 2 & \cdots & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & \cdots & 2 \\
2 & \cdots & 2 & 2 & \cdots & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & \cdots & 2 \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0
\end{array}\right].$$

This matrix presentation makes use of the affine identification $\mathrm{Spin}^c(\mathcal{Z}_{p,k_p}) \cong H^2(\mathcal{Z}_{p,k_p}) \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$, where $(i,j) \in \mathbb{Z}_p \oplus \mathbb{Z}_p$ is such that $|i|, |j| \leq \frac{p-1}{2}$. There is an indeterminacy present that must be discussed. In Appendix 3.5, the calculation of the correction terms for $Y = S^3_{p^2+p}(J\#J\#T_{p,p+1})$ (with $J = \#_{k_p} D$) is done in a way that forgets the explicit identification of $\mathrm{Spin}^c(Y) \cong H^2(Y) \cong \mathbb{Z}_p \oplus \mathbb{Z}_{p+1}$. Thus, we lose track of the difference between

j and $-j$ in $\mathbb{Z}_{n+1}$ and between i and $-i$ in $\mathbb{Z}_n$. As a consequence, when regarding the matrix above, we must consider it only up to horizontal reflection about the central column and vertical reflection about the central row. This indeterminacy is inconsequential in what follows. In particular, the following corollary holds.

Corollary 3.6.2. The manifold $\mathcal{Z}_{p,k_p}$ has precisely $2p - 2k_p - 1$ vanishing correction terms. Therefore, $\mathcal{K}_{p,k_p}$ is not smoothly doubly slice. Moreover, the $\mathcal{K}_{p,k_p}$ are nontrivial in $\mathcal{C}_{\mathcal{D}}$.

Proof. The corresponding $(p \times p)$–matrix for $\mathcal{D}(L(p,1)\#L(p,-1))$ has zeros along the two (orthogonal) diagonals and non-integer rational numbers elsewhere. It is easy to see that precisely $2k_p$ of these $2p-1$ vanishing entries will be lowered by a nonzero amount. Since any entry off these diagonals is not an even integer, no new zeros will be created. Therefore, $\mathcal{Z}_{p,k_p}$ has precisely $2p-2k_p-1$ vanishing correction terms. By Theorem 3.2.2, this implies that the $\mathcal{K}_{p,k_p}$ are not smoothly doubly slice. In fact, by Proposition 3.3.10, this implies that each $\mathcal{K}_{p,k_p}$ is not even smoothly stably doubly slice, since $\det(\mathcal{K}_{p,k_p}) = p^2$. Therefore, each $\mathcal{K}_{p,k_p}$ represents a nontrivial element in $\mathcal{C}_{\mathcal{D}}$. $\square$

3.6.1 Notation and set-up

Let $X = S^3_n(K)$, and let $[\mathfrak{s}_i] \in \mathrm{Spin}^c(X)$ be the enumeration of $\mathrm{Spin}^c(X)$

introduced in Subsection 3.3.4. Then we have the following decomposition:

$$HF^{\infty}(X) = \bigoplus_{i=0}^{n-1} HF^{\infty}(X, \mathfrak{s}_i) = \bigoplus_{i=0}^{n-1} \mathcal{T}_i^{\infty}(X).$$

Note that here and throughout, subscripts will correspond to the labelings of Spin^c structures on the manifolds. Theorem 3.4.6 implies that, for any $x \in \mathcal{T}_i^{\infty}(X)$,

$$gr(x) \equiv d(L(n,1), i) \pmod 2$$

for all $i \in \mathbb{Z}_n$. Let $\bar{x}_i^{\infty}$ denote the element in $\mathcal{T}_i^{\infty}(X)$ such that

$$gr(\bar{x}_i^{\infty}) = d(L(n,1), i).$$

Let $Y = X_{-n-1}(K)$ for a null homologous knot K in X, and let $[\mathfrak{s}_i, \mathfrak{s}_j] \in \mathrm{Spin}^c(Y)$ be the enumeration of $\mathrm{Spin}^c(Y)$, as in Subsection 3.3.4. This gives the following decomposition:

$$HF^{\infty}(Y) = \bigoplus_{i=0}^{n-1} \bigoplus_{j=0}^{n} HF^{\infty}(Y, [\mathfrak{s}_i, \mathfrak{s}_j]) = \bigoplus_{i=0}^{n-1} \bigoplus_{j=0}^{n} \mathcal{T}_{i,j}^{\infty}(Y).$$

Let $F^{\infty}_{W_1,[\mathfrak{s}_i,\mathfrak{t}_m]} : HF^{\infty}(X, \mathfrak{s}_i) \to HF^{\infty}(Y, [\mathfrak{s}_i, \mathfrak{s}_j])$ be the map induced by $(W_1, [\mathfrak{s}_i, \mathfrak{t}_m])$, as in Subsection 3.4.2. Since W_1 is negative definite, we can conclude (see [OS03a]) that $F^{\infty}_{W_1,\mathfrak{t}}$ is an isomorphism for all $\mathfrak{t} \in \mathrm{Spin}^c(W_1)$. Furthermore,

$$gr\left(F^{\infty}_{W_1,[\mathfrak{s}_i,\mathfrak{t}_m]}\right) = \frac{(n+1) - (2m + (n+1))^2}{4(n+1)}$$

for each $i \in \mathbb{Z}_n$. In general, if F is any graded map between graded abelian groups, we denote the grading shift of F by $gr(F)$.

Lemma 3.6.3. For all $i \in \mathbb{Z}_n$ and $j \in \mathbb{Z}_{n+1}$, let y be any element in $\mathcal{T}^{\infty}_{i,j}(Y)$, then

$$gr(y) \equiv gr(L(n,1),i) - gr(L(n+1,1),j) \pmod 2.$$

Proof. The fact that $F^{\infty}_{W_1,[\mathfrak{s}_i,\mathfrak{t}_m]}$ is an isomorphism, and the labeling of Spin^c structures, implies that $F^{\infty}_{W_1,[\mathfrak{s}_i,\mathfrak{t}_m]}(x_i^{\infty}) \subset \mathcal{T}^{\infty}_{i,j}$ if and only if $m \equiv j \pmod{n+1}$. Let $m = -j$, then, since all elements in $\mathcal{T}^{\infty}_{i,j}$ can be obtained from each other by translation by U,

$$\begin{aligned} gr(y) &\equiv gr\left(F^{\infty}_{W_1,[\mathfrak{s}_i,\mathfrak{t}_{-j}]}(\bar{x}_i^{\infty})\right) \pmod 2 \\ &\equiv gr(\bar{x}_i^{\infty}) + gr(F^{\infty}_{W_1,[\mathfrak{s}_i,\mathfrak{t}_{-j}]}) \\ &\equiv d(L(n,1),i) - d(L(n+1,1),j) \end{aligned}$$

□

Let $\bar{y}^{\infty}_{i,j}$ denote the element in $\mathcal{T}^{\infty}_{i,j}(Y)$ satisfying

$$gr(\bar{y}^{\infty}_{i,j}) = d(L(n,1),i) - d(L(n+1,1),j).$$

Using this notation, we gain a precise understanding of the map

$$F_1^{\infty} = \sum_{\mathfrak{t} \in \mathrm{Spin}^c(W_1)} F^{\infty}_{W_1,\mathfrak{t}},$$

given by the following lemma. Note that F_1^{∞} is not a well-defined map to $HF^{\infty}(Y)$, since its image will generally consist of infinite sums of elements in $HF^{\infty}(Y)$. The important fact for us is that all but finitely many of the terms will have coefficients that are large powers of U.

Lemma 3.6.4. Let the $\bar{x}_i^\infty$ and $\bar{y}_{i,j}^\infty$ be defined as above. Then, for all $i \in \mathbb{Z}_n$,

$$F_1^\infty(\bar{x}_i^\infty) = (\bar{y}_{i,1}^\infty + \bar{y}_{i,2}^\infty + \cdots + \bar{y}_{i,n}^\infty) + U(\bar{y}_{i,1}^\infty + \bar{y}_{i,n}^\infty) + U^2(\bar{y}_{i,2}^\infty + \bar{y}_{i,n-1}^\infty) + \cdots ,$$

where the expression continues indefinitely with increasing positive powers of U as coefficients.

Proof. The proof of this lemma is a simple examination of $gr(F^\infty_{W_1,[\mathfrak{s}_i,\mathfrak{t}_m]})$ as m varies over the integers. The powers of U in the tail follow a growth pattern that depends quadratically on n in a simple way, but will not be relevant in what follows. $\square$

Continuing, let Z be obtained from Y by blowing down a meridian, as in Subsection 3.3.3, let W_2 be the induced cobordism, and let $[\mathfrak{s}_i, \mathfrak{s}_k] \in \mathrm{Spin}^c(Z)$ be the enumeration of $\mathrm{Spin}^c(Z)$, as in Subsection 3.3.4. This gives the following decomposition:

$$HF^\infty(Z) = \bigoplus_{i=0}^{n-1}\bigoplus_{k=0}^{n-1} HF^\infty(Z, [\mathfrak{s}_i, \mathfrak{s}_k]) = \bigoplus_{i=0}^{n-1}\bigoplus_{k=0}^{n-1} \mathcal{T}^\infty_{i,k}(Z).$$

Let $F^\infty_{W_2,[\mathfrak{s}_i,\mathfrak{r}_m]} : HF^\infty(Y, [\mathfrak{s}_i, \mathfrak{s}_j]) \to HF^\infty(Z, [\mathfrak{s}_i, \mathfrak{s}_k])$ be the map induced by $(W_2, [\mathfrak{s}_i, \mathfrak{r}_m])$. Since W_2 is negative definite, we can conclude that $F^\infty_{W_2,\mathfrak{r}}$ is an isomorphism for all $\mathfrak{r} \in \mathrm{Spin}^c(W_2)$. Furthermore,

$$gr\left(F^\infty_{W_2,[\mathfrak{s}_i,\mathfrak{r}_m]}\right) = \frac{n(n+1) - (2m + n(n+1))^2}{4n(n+1)}$$

for each $i \in \mathbb{Z}_n$ and $j \in \mathbb{Z}_{n+1}$.

Lemma 3.6.5. Let z be any element in $\mathcal{T}^{\infty}_{i,k}(Z)$, then

$$gr(z) \equiv gr(L(n,1),i) - gr(L(n,1),k) \pmod 2$$

for all $i \in \mathbb{Z}_n$ and $k \in \mathbb{Z}_n$.

Proof. This proof is identical to that of Lemma 3.6.3. $\square$

Let $\bar{z}^{\infty}_{i,k}$ denote the element in $\mathcal{T}^{\infty}_{i,k}(Z)$ satisfying

$$gr(\bar{z}^{\infty}_{i,k}) = d(L(n,1),i) - d(L(n,1),j).$$

Using this notation, we gain a precise understanding of the map

$$F^{\infty}_{2} = \sum_{\mathfrak{r} \in \mathrm{Spin}^c(W_2)} F^{\infty}_{W_2,\mathfrak{r}},$$

in an analogous way to Lemma 3.6.4. From this point on, we will index $H_1(X) \cong \mathbb{Z}_n$, $H_1(Y) \cong \mathbb{Z}_n \oplus \mathbb{Z}_{n+1}$, and $H_1(Z) \cong \mathbb{Z}_n \oplus \mathbb{Z}_n$ by $i, (i,j)$, and (i,k) (respectively), such that $-\frac{n-1}{2} \leq i,k \leq \frac{n-1}{2}$ and $-\frac{n+1}{2} \leq j \leq \frac{n-1}{2}$.

Lemma 3.6.6. Let the $\bar{y}^{\infty}_{i,j}$ and $\bar{z}^{\infty}_{i,k}$ be defined as above. Then, for all $i \in \mathbb{Z}_n$,

$$\begin{aligned} F^{\infty}_{2}(\bar{y}^{\infty}_{i,0}) &= \bar{z}^{\infty}_{i,0} + U(\bar{z}^{\infty}_{i,1} + \bar{z}^{\infty}_{i,n-1}) + U^5(\bar{z}^{\infty}_{i,2} + \bar{z}^{\infty}_{i,n-2}) + \cdots, \\ F^{\infty}_{2}(\bar{y}^{\infty}_{i,j}) &= \bar{z}^{\infty}_{i,j-1} + \bar{z}^{\infty}_{i,j} + U(\bar{z}^{\infty}_{i,j+1}) + U^3(\bar{z}^{\infty}_{i,j-2}) + \cdots, \end{aligned}$$

if $|j| = 1$, and

$$F^{\infty}_{2}(\bar{y}^{\infty}_{i,j}) = \bar{z}^{\infty}_{i,j-1} + \bar{z}^{\infty}_{i,j} + U(\bar{z}^{\infty}_{i,j-2} + \bar{z}^{\infty}_{i,j+1}) + U^3(\bar{z}^{\infty}_{i,j-3} + \bar{z}^{\infty}_{i,j+2}) + \cdots,$$

for $|j| > 1$. The expressions continue indefinitely with increasing positive powers of U as coefficients.

Proof. This proof is the same as that of Lemma 3.6.4. $\square$

Let $\pi : HF^{\infty}(M, \mathfrak{s}) \to HF^{+}(M, \mathfrak{s})$, be the natural projection map. Let $\bar{x}_i^+ = \pi(\bar{x}_i^{\infty})$, and define $\bar{y}_{i,j}^+$ and $\bar{z}_{i,k}^+$ similarly. Analogous to the discussion above, we have the following decomposition:

$$HF^+(X)/HF_{red}(X) = \bigoplus_{i=-\frac{n-1}{2}}^{\frac{n-1}{2}} \mathcal{T}_i^+(X),$$

as well as similar decompositions corresponding to Y and Z. Note that we are not claiming that $\bar{x}_i^+$ is nonzero in $\mathcal{T}_i^+(X)$. Similarly, it may be that $\bar{y}_{i,j}^+$ and the $\bar{z}_{i,k}^+$ vanish. Define

$$F_1^+ = \sum_{\mathfrak{t} \in \mathrm{Spin}^c(W_1)} F_{W_1,\mathfrak{t}}^+,$$

and

$$F_2^+ = \sum_{\mathfrak{r} \in \mathrm{Spin}^c(W_2)} F_{W_2,\mathfrak{r}}^+.$$

3.6.2 Proof of Proposition 3.6.1

With this notational set-up, we recall that the triad (X, Y, Z) introduced in Section 3.3 induces certain long exact sequence (discussed in Section 3.4), which will be used below in the proof of Proposition 3.6.1.

Let $J = \#_{k_p} D$, so $X = S_p^3(J \# J)$, $Y = S_{p^2+p}^3(J \# J \# T_{p,p+1})$, and $Z = \mathcal{Z}_{p,k_p} = \Sigma_2(I_{\#_{k_p} D, p})$. The calculations made in Appendix 3.5 give us the correction terms for X and Y. In particular, Lemma 3.5.2 tells us that $\mathcal{D}(L(p,1)) - \mathcal{D}(X)$ is given by

$$\vec{w} = \{0, \ldots, 0, 2, 2, 4, 4, \ldots, 2k_p{-}2, 2k_p{-}2, 2k_p, 2k_p, 2k_p, 2k_p{-}2, 2k_p{-}2, \ldots, 4, 4, 2, 2, 0, \ldots, 0\},$$

where $2w_i$ is the value of the i^{th} coordinate of $\vec{w}$ for $i \in \mathbb{Z}$ with our symmetric labeling.

Let $x_i^\infty = U^{w_i}\bar{x}_i^\infty$, and let $\pi(x_i^\infty) = x_i^+$. It follows that x_i^+ is the element of lowest grading in $\mathcal{T}_i^+(X)$, i.e., $gr(x_i^+) = d(X, \mathfrak{s}_i)$. Similarly, by Corollary 3.5.6, $\mathcal{D}(L(n,1)\#L(n+1,-1)) - \mathcal{D}(Y)$ is given by the matrix $\mathcal{M} = (2m_{i,j})$, which has the following form:

$$\begin{bmatrix}
0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
2 & \cdots & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & \cdots & 2 & 2 & 2 & \cdots & 2 \\
2 & \cdots & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & \cdots & 2 & 2 & 2 & \cdots & 2 \\
4 & \cdots & 4 & 4 & 4 & 2 & 0 & 0 & 0 & 0 & 0 & 4 & 4 & 4 & 4 & \cdots & 4 & 4 & 4 & \cdots & 4 \\
4 & \cdots & 4 & 4 & 4 & 4 & 2 & 0 & 0 & 0 & 0 & 4 & 4 & 4 & 4 & \cdots & 4 & 4 & 4 & \cdots & 4 \\
\vdots & & \vdots & \vdots & \vdots & & & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
2k_p & \cdots & 2k_p & 2k_p & 2k_p & \cdots & 2k_p & 2k_p & 2 & 0 & 0 & 2k_p & 2k_p & 2k_p & 2k_p & \cdots & 2k_p & 2k_p & 2k_p & \cdots & 2k_p \\
2k_p & \cdots & 2k_p & 2k_p & 2k_p & \cdots & 2k_p & 2k_p & 2k_p & 2 & 0 & 2 & 2k_p & 2k_p & 2k_p & \cdots & 2k_p & 2k_p & 2k_p & \cdots & 2k_p \\
2k_p & \cdots & 2k_p & 2k_p & 2k_p & \cdots & 2k_p & 2k_p & 2k_p & 2k_p & 0 & 0 & 2 & 2k_p & 2k_p & \cdots & 2k_p & 2k_p & 2k_p & \cdots & 2k_p \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \ddots & \ddots & \ddots & & & \vdots & \vdots & \vdots & & \vdots \\
4 & \cdots & 4 & 4 & 4 & \cdots & 4 & 4 & 4 & 4 & 0 & 0 & 0 & 0 & 2 & 4 & 4 & 4 & 4 & \cdots & 4 \\
4 & \cdots & 4 & 4 & 4 & \cdots & 4 & 4 & 4 & 4 & 0 & 0 & 0 & 0 & 0 & 2 & 4 & 4 & 4 & \cdots & 4 \\
2 & \cdots & 2 & 2 & 2 & \cdots & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & \cdots & 2 \\
2 & \cdots & 2 & 2 & 2 & \cdots & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & \cdots & 2 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0
\end{bmatrix}$$

Note that the values in the i^{th} row of $\mathcal{M}$ are bounded above by $2w_i$. (Remember that the rows are labeled by $\mathbb{Z}_n$ symmetrically about zero, and the columns are labeled by $\mathbb{Z}_{n+1}$ by $j \in [-\frac{n+1}{2}, \frac{n-1}{2}]$). We remark again that the calculation given in the proof of Corollary 3.5.6 introduces an indeterminacy regarding our presentation of the correction terms. Namely, we cannot distinguish between i and $-i$ and j and $-j$ in the present labeling. This indeterminacy is merely notational and will not affect the results.

Let $y_{i,j}^{\infty} = U^{m_{i,j}} \bar{y}_{i,j}^{\infty}$, and let $y_{i,j}^{+} = \pi(y_{i,j}^{\infty})$. It follows that $y_{i,j}^{+}$ is the element of lowest grading in $\mathcal{T}_{i,j}^{+}(Y)$, i.e., $gr(y_{i,j}^{+}) = d(Y, [\mathfrak{s}_i, \mathfrak{s}_j])$. With this notational set-up, we can prove the following lemma about the map $F_1^{+} : HF^{+}(X) \to HF^{+}(Y)$.

Lemma 3.6.7. Let $x_i^{+} \in \mathcal{T}_i(X)$ and $y_{i,j}^{+} \in \mathcal{T}_{i,j}^{+}(Y)$ be elements of lowest grading in their respective towers. Then,

$$F_1^{+}(x_i^{+}) = \sum_{j \in \mathcal{I}_i} y_{i,j}^{+},$$

where $\mathcal{I}_i = \{j \neq 0 \ : \ m_{i,j} = w_i\}$.

Proof. By Lemma 3.6.4 we have that

$$F_1^{\infty}(\bar{x}_i^{\infty}) = \sum_{j \neq 0} \bar{y}_{i,j}^{\infty} + \mathcal{U}(\bar{y}_{i,j}^{\infty}),$$

where $\mathcal{U}(\bar{y}_{i,j}^{\infty})$ represents the terms that are positive U–translates of the $\bar{y}_{i,j}^{\infty}$. By U–equivariance, we have

$$F_1^{\infty}(x_i^{\infty}) = U^{w_i} F_1^{\infty}(\bar{x}_i^{\infty}) = \sum_{j \neq 0} U^{w_i} \bar{y}_{i,j}^{\infty} + U^{w_i} \mathcal{U}(\bar{y}_{i,j}^{\infty}).$$

Since F_1 commutes with the natural projection π (which is U–equivariant), we see that

$$F_1^+(x_i^+) = \pi(F_1^\infty(x_i^\infty)) = \sum_{j\neq 0} U^{w_i}\pi(\bar{y}_{i,j}^\infty) = \sum_{j\neq 0} U^{w_i}\bar{y}_{i,j}^+,$$

where the tail has vanished, by U–equivariance. By definition, we have $U^{w_i}\bar{y}_{i,j}^+ = U^{w_i - m_{i,j}}y_{ij}^+$, and this term will be nonzero if and only if $w_i \leq m_{i,j}$. This can only happen if $w_i = m_{i,j}$, since, as we noticed above, $m_{i,j} \leq w_i$. $\square$

Note that $|\mathcal{J}_i| \geq \frac{p+1}{2}$ for each i; so, in particular, $F_1^+(x_i^+)$ is a linear combination of at least $\frac{p+1}{2}$ terms for each i. One consequence of this is that $y_{i,j}^+$ is not in the image of F_1^+ for any i, j.

Let $z_{i,j}^+$ denote the element of lowest grading in $\mathcal{T}_{i,j}^+(Z)$. We know by U–equivariance that

$$F_{W_2,[\mathfrak{s}_i,\mathfrak{s}_k]}(y_{i,j}^+) = U^{c_{i,k}}z_{i,k}^+$$

for some nonnegative integer $c_{i,k}$. If we can show that $c_{i,j} = 0$ for all i, k, we will have proved Proposition 3.6.1, because we will have shown that $z_{i,k}^\infty = U^{m_{i,k}}\bar{z}_{i,k}^\infty$. This is accomplished by the following lemma. Recall the natural inclusion map $\hat{\imath} : \widehat{HF}(Z) \to HF^+(Z)$.

Lemma 3.6.8. Let $z_{i,k}^+$ be the element of lowest grading in $\mathcal{T}_{i,k}^+(Z)$, and let $y_{i,k}^+$ be the element of lowest grading in $\mathcal{T}_{i,k}^+(Y)$. Then,

$$gr(z_{i,k}^+) = gr\left(F_{W_2,[\mathfrak{s}_i,\mathfrak{t}_k]}^+(y_{i,k}^+)\right).$$

Proof. Let $\hat{z} \in \widehat{HF}(Z)$ such that $\hat{\imath}(\hat{z}) = z^+_{i,k}$. By Theorem 3.4.5, we know that $\widehat{F}_3(\hat{z}) = 0$. (Recall that $-\overline{W_3}$ is induced by $(-p)$–surgery on a knot of genus $2k_p$ with $p > 4p_k - 1$, see Subsection 3.3.5.) By the exactness at $\widehat{HF}(X)$, there exists some $\hat{y} \in \widehat{HF}(Y)$ such that $\widehat{F}_2(\hat{y}) = \hat{z}$. Now, $\hat{y}$ may not be homogeneous, so write $\hat{y} = \sum_a \hat{y}_a$, where each $\hat{y}_a$ is homogeneous and in $\widehat{HF}(Y', [\mathfrak{s}_i, \mathfrak{s}_{j_a}])$. By Lemma 3.5.8, we know that for each a, $gr(\hat{y}_a) \leq gr(y^+_{i,j_a})$. So, we have

$$\begin{aligned} gr\left(\widehat{F}_{W_2,[\mathfrak{s}_i,\mathfrak{t}_{m_a}]}(\hat{y}_a)\right) &= gr(\hat{y}_a) + gr\left(\widehat{F}_{W_2,[\mathfrak{s}_i,\mathfrak{t}_{m_a}]}\right) \\ &\leq gr(y^+_{i,j_a}) + gr\left(F^+_{W_2,[\mathfrak{s}_i,\mathfrak{t}_{m_a}]}\right) \\ &= gr\left(F_{W_2^+,[\mathfrak{s}_i,\mathfrak{t}_{m_a}]}(y^+_{i,j_a})\right) \\ &\leq gr\left(F_{W_2^+,[\mathfrak{s}_i,\mathfrak{t}_k]}(y^+_{i,k})\right), \end{aligned}$$

where the last inequality follows from the fact that $gr\left(F_{W_2^+,[\mathfrak{s}_i,\mathfrak{t}_m]}(y^+_{i,j})\right)$ is maximized when by j with $|j| = k$. (Note that $j_a \equiv k \pmod{p}$.) Since

$$gr(\hat{z}) \leq \max_a gr\left(\widehat{F}_{W_2,[\mathfrak{s}_i,\mathfrak{t}_{m_a}]}(\hat{y}_a)\right),$$

we have

$$gr(\hat{z}) \leq gr\left(F_{W_2^+,[\mathfrak{s}_i,\mathfrak{t}_k]}(y^+_{i,k})\right).$$

This implies the desired equality once we recall that

$$gr(\hat{z}) = gr(z^+_{i,k}) \geq gr\left(F_{W_2^+,[\mathfrak{s}_i,\mathfrak{t}_k]}(y^+_{i,k})\right),$$

by U–equivariance. $\square$

3.7 Proof of Theorem F

In this subsection, we give a reformulation of one of the invariants introduced in [GRS08] for the study of double concordance of knots and use it to find an infinitely generated subgroup in $\ker \psi_{\mathfrak{D}}$.

Let A be a finite abelian group, so A can be written as the product of cyclic groups. Let $r_{p,k}(A)$ denote the number of copies of $\mathbb{Z}_{p^k}$ in the decomposition of A. Let $r_p(A) = \sum_{k=1}^{\infty} r_{p,k}(A)$. In other words, any generating set for A must contain at least $r_p(A)$ elements of order p^k for some $k \in \mathbb{N}$.

The following definition differs from [GRS08] only in the use of $r_p(A)$.

Definition 3.7.1. Let K be a knot in S^3 and let $p \in \mathbb{N}$ be a positive prime. Let $M = \Sigma_2(K)$. Fix an affine identification between $\mathrm{Spin}^c(M)$ and $A = H^2(M;\mathbb{Z})$ such that the unique spin structure $\mathfrak{s}_0$ gets identified with zero in A. Let $\mathcal{G}_p$ denote the collection of all subgroups of A of order p. Define

$$\mathfrak{D}_p(K) = \min \left\{ \left| \sum_{H \in \mathcal{G}_p} n_H S_H(d(M)) \right| \; : \; \begin{array}{c} n_H \geq 0 \text{ for all } H, \\ \text{at least } r_p(A) \text{ of the } n_H \text{ are nonzero} \end{array} \right\}$$

if p divides $\det(K)$ and

$$\mathfrak{D}_p(K) = 0$$

otherwise, where $S_H(d(M)) = \sum_{h \in H} d(M,h)$.

The proof of the following theorem is essentially given in [GRS08], but is formulated here for double concordance.

Theorem 3.7.2. Let $K \subset S^3$ be a knot and $p \in \mathbb{N}$ a positive prime. If there is a positive $n \in \mathbb{N}$ such that $\#_n K$ is smoothly doubly slice, then $\mathfrak{D}_p(K) = 0$.

Proof. Suppose that $J = \#_n K$ is smoothly doubly slice. Let $N = \Sigma_2(J) = \#_n \Sigma_2(K)$. The identification of $\mathrm{Spin}^c(\Sigma_2(K))$ with A gives an identification of $\mathrm{Spin}^c(N)$ with A^n. By Theorem 3.2.2, there exists subgroups G and H in A^n such that $G \oplus H = A^n$, $G \cong H$, and $G \cap H = \{0\}$.

Assume without loss of generality that p divides $\det(K)$, and let $r = r_p(A)$. Projection onto the first coordinate $\pi : A^n \to A$ is onto, so $\pi(G) + \pi(H) = A$. Let $a_1, \ldots, a_r$ be linearly independent generators of A of p–power order such that $\pi^{-1}(a_i) \cap (G \cup H)$ is nonempty. Let $g_i' \in \pi^{-1}(a_i) \cap (G \cup H)$, then $|g_i'| = p^{k_i} q$ for some positive $q \in \mathbb{Z}$ relatively prime to p. Let $g_i = q p^{k_i - 1} g_i'$. Then $\{g_1, \ldots, g_r\}$ is a collection elements of order p in $G \cup H$. Furthermore, the elements of $\{\pi(g_1), \ldots, \pi(g_r)\}$ are linearly independent in A, so, as elements of $\mathcal{G}_p$, $\langle g_i \rangle = \langle g_j \rangle$ if and only if $i = j$. Let $g_i = (g_i^1, \ldots, g_i^n)$ for $i = 1, \ldots, r$.

By Theorem 3.2.2, $d(N, x) = 0$ for all $x \in G \cup H$. Let $f : A \to \mathbb{Q}$ be given by $f(x) = d(N, x)$, and let $f^{(n)} : A^n \to \mathbb{Q}$ be given by $f(x_1, \ldots, x_n) =$

$f(x_1) + \cdots + f(x_n)$. Since $\langle g_i \rangle < G \cup H$, we have

$$\begin{aligned}
\sum_{m=0}^{p-1} f^{(n)}(mg_i) = 0 \quad &\implies \quad \sum_{m=0}^{p-1} \sum_{j=1}^{n} f(mg_i^j) = 0 \\
&\implies \quad \sum_{j=1}^{n} \sum_{m=0}^{p-1} f(mg_i^j) = 0 \\
&\implies \quad \sum_{j=1}^{n} S_{\langle g_i^j \rangle}(f) = 0 \\
&\implies \quad \sum_{j=1}^{n} S_{\langle g_i^j \rangle}(d(N)) = 0
\end{aligned}$$

Since $\langle g_i^j \rangle \in \mathcal{G}_p$ for each j,

$$\sum_{j=1}^{n} S_{\langle g_i^j \rangle}(d(N)) = \sum_{H \in \mathcal{G}_p} n_H S_H(d(N)),$$

with at least one n_H nonzero (since at least g_i^1 is nontrivial). For each $j = 1, \ldots, r$, we get a similar linear combination, and, since the g_j^1 are independent, each linear combination is nontrivial on a distinct element of $\mathcal{G}_p$. Summing, we get

$$\sum_{i=1}^{r} \sum_{j=1}^{n} S_{\langle g_i^j \rangle}(d(N)) = \sum_{H \in \mathcal{G}_p} n_H S_H(d(N)),$$

where at least r or the n_H are nonzero. It follows that $\mathcal{D}_p(K) = 0$, as desired.

$\square$

To prove Theorem F, we will need to understand $S_G(f)$ for each subgroup G of $\mathbb{Z}_p \oplus \mathbb{Z}_p$. Let $G_\star = \langle (1,1) \rangle$ and let $G_a = \langle (a, a+1) \rangle$ for $a \in \mathbb{Z}_p$. Then, together, $G_\star$ and the G_a represent the $p+1$ distinct subgroups of $\mathbb{Z}_p \oplus \mathbb{Z}_p$.

First let's consider $Z = L(p,1)\#L(p,-1)$ for a positive prime p. We saw in Subsection 3.3.4 that we have an affine identification $[\mathfrak{s}_i, \mathfrak{s}_j] \sim (i,j)$ between $\mathrm{Spin}^c(Z)$ and $Z_p \oplus \mathbb{Z}_p$.

Let $f : H_1(Z) \to \mathbb{Q}$ be given by $f(x) = d(Z, [\mathfrak{s}_i, \mathfrak{s}_j])$, where $[\mathfrak{s}_i, \mathfrak{s}_j] \sim x$ is the given affine identification. It is possible to check using Equation 3.1 that

$$S_a^{\mathrm{lens}} = S_{G_a}(f) = \begin{cases} \frac{(p-1)(p+1)}{6} & \text{if } a = 0, \\ -\frac{(p-1)(p+1)}{6} & \text{if } a = p-1, \\ 0 & \text{if } a = \star, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, by Proposition 3.6.1, we know that

$$\mathcal{D}(L(n,1)\#L(n,-1)) - \mathcal{D}(\mathcal{Z}_{p,k_p})$$

is given by

$$\begin{bmatrix}
0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
2 & \cdots & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & \cdots & 2 & 2 & \cdots & 2 \\
2 & \cdots & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & \cdots & 2 & 2 & \cdots & 2 \\
4 & \cdots & 4 & 4 & 2 & 0 & 0 & 0 & 0 & 0 & 4 & 4 & 4 & 4 & \cdots & 4 & 4 & \cdots & 4 \\
4 & \cdots & 4 & 4 & 4 & 2 & 0 & 0 & 0 & 0 & 4 & 4 & 4 & 4 & \cdots & 4 & 4 & \cdots & 4 \\
\vdots & & \vdots & \vdots & & & \ddots & \ddots & \ddots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
2k_p & \cdots & 2k_p & 2k_p & \cdots & 2k_p & 2k_p & 2 & 0 & 0 & 2k_p & 2k_p & 2k_p & 2k_p & \cdots & 2k_p & 2k_p & \cdots & 2k_p \\
2k_p & \cdots & 2k_p & 2k_p & \cdots & 2k_p & 2k_p & 2k_p & 2 & 0 & 2 & 2k_p & 2k_p & 2k_p & \cdots & 2k_p & 2k_p & \cdots & 2k_p \\
2k_p & \cdots & 2k_p & 2k_p & \cdots & 2k_p & 2k_p & 2k_p & 2k_p & 0 & 0 & 2 & 2k_p & 2k_p & \cdots & 2k_p & 2k_p & \cdots & 2k_p \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \ddots & \ddots & \ddots & & & \vdots & \vdots & & \vdots \\
4 & \cdots & 4 & 4 & \cdots & 4 & 4 & 4 & 4 & 0 & 0 & 0 & 0 & 2 & 4 & 4 & 4 & \cdots & 4 \\
4 & \cdots & 4 & 4 & \cdots & 4 & 4 & 4 & 4 & 0 & 0 & 0 & 0 & 0 & 2 & 4 & 4 & \cdots & 4 \\
2 & \cdots & 2 & 2 & \cdots & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & \cdots & 2 \\
2 & \cdots & 2 & 2 & \cdots & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & \cdots & 2 \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0
\end{bmatrix}.$$

Let $S'_G = \sum_{g \in G} \mathcal{M}_g$, where $\mathcal{M}$ is the above matrix and $\mathcal{M}_g = \mathcal{M}_{i,j}$, if $g = (i,j) \in \mathbb{Z}_p$. Then, we see that

$$S'_{G_a} = \begin{cases} 2k(p-3)+4 & \text{if } a = 0, \\ 0 & \text{if } a = p-1, \\ 0 & \text{if } a = \star, \\ \text{(large positive number)} & \text{otherwise.} \end{cases}$$

It follows that the pertinent sums for $\mathcal{Z}_{p,k_p}$ are given by $S^{\mathcal{Z}_{p,k_p}}_{G_a}(f) = S^{\text{lens}}_a - S'_{G_a}$. So,

$$S^{\mathcal{Z}_{p,k_p}}_{G_a} = \begin{cases} \frac{(p-1)(p+1)}{6} - (2k(p-3)+4) & \text{if } a = 0, \\ -\frac{(p-1)(p+1)}{6} & \text{if } a = p-1, \\ 0 & \text{if } a = \star, \\ \text{(large negative number)} & \text{otherwise.} \end{cases}$$

The upshot is that $S^{\mathcal{Z}_{p,k_p}}_{G_a}$ will be strictly negative for all $a \neq \star$ if and only if

$$\frac{(p-1)(p+1)}{6} - (2k(p-3)+4) < 0.$$

The left side will be negative if $k \geq \frac{p+5}{12}$. As we saw above in Subsection 3.3.5, we will let $k_p = \left\lceil \frac{p+6}{12} \right\rceil$, which will satisfy this condition. Now we can prove the following, recalling our set-up from Section 3.3.

Proposition 3.7.3. Let $\mathcal{K}_{p,k_p} = I_{\#_{2k_p} J,p}$, where J is $T_{2,3}$ or D, and where $k_p = \left\lceil \frac{p+6}{12} \right\rceil$. Then,

1. No knot in the span (under connected sum) of the $\mathcal{K}_{p,k_p}$ is smoothly doubly slice.

2. Each of the $\mathcal{K}_{p,k_p}$ has order greater than two in $\mathcal{C}_{\mathcal{D}}$.

3. The collection $\{\mathcal{K}_{p,k_p}\}$ forms a basis for an infinitely generated subgroup of $\mathcal{C}_{\mathcal{D}}$.

Note that this is independent of the indeterminacies $i \leftrightarrow -i$ and $j \leftrightarrow -j$ discussed earlier. Notice also that Example 3.3.9 illustrates why we cannot claim that the $\mathcal{K}_{p,k_p}$ have infinite order in $\mathcal{C}_{\mathcal{D}}$.

Proof. By Corollary 3.6.2, we know that each of these knots is nontrivial in $\mathcal{C}_{\mathcal{D}}$. First, we show (1). The discussion preceding this proposition shows that the Grigsby-Ruberman-Strle invariant $\mathcal{D}_p$ is nonzero for $\mathcal{K}_{p,k_p}$. This follows because, for these knots, $S_{G_a}^{\mathcal{X}_{p,k_p}}$ is nonnegative for only one subgroup of $\mathbb{Z}_p \oplus \mathbb{Z}_p$. Since the condition on $\mathcal{D}_p$ states that n_G must be nonzero for at least two distinct subgroups G, the sum $\sum_{G \in \mathcal{G}_p} n_G S_G(M)$ will always be nonzero. By Theorem 3.7.2, this shows that $\#_a \mathcal{K}_{p,k_p}$ is not doubly slice for all $a \in \mathbb{N}$. By Proposition 3.3.10, $\mathcal{K}_{p,k_p} \# \mathcal{K}_{p,k_p}$ is nontrivial in $\mathcal{C}_{\mathcal{D}}$, since $\mathcal{A}_p$ is rank 4 and not hyperbolic for these knots.

Suppose that

$$\mathcal{K} = \left(\#_{n_{p_1}} \mathcal{K}_{p_1,k_{p_1}}\right) \# \left(\#_{n_{p_2}} \mathcal{K}_{p_2,k_{p_2}}\right) \# \cdots \# \left(\#_{n_{p_m}} \mathcal{K}_{p_m,k_{p_m}}\right).$$

Since the p_i are all distinct primes, we get that $\mathcal{D}_{p_i}(K) = \mathcal{D}_{p_i}(\#_{n_i} \mathcal{K}_{p_i,k_{p_i}})$. It is easy to see that, for the knots in question, $\mathcal{D}_{p_i}(\#_{n_i} \mathcal{K}_{p_i,k_{p_i}}) \neq 0$,since $S_{G_a}^{\mathcal{Z}_{p_i,k_{p_i}}}$ is always nonpositive and strictly negative away from a single metabolizer. It follows that $\mathcal{D}_{p_i}(\mathcal{K}) \neq 0$. This proves that K is not doubly slice, and if any of the n_{p_i} are less than 3, then $\mathcal{K}$ is nontrivial in $\mathcal{C}_{\mathcal{D}}$. $\square$

By Corollary 3.3.5, each member of $\{\mathcal{K}_{p,k_p}\}$ is topologically doubly slice. It follows that these knots generate an infinitely generated subgroup of $\ker \psi_{\mathcal{D}}$ that consists of knots that are not smoothly doubly slice. This proves Theorem F.

Bibliography

[AK10] Tetsuya Abe and Kengo Kishimoto, *The dealternating number and the alternation number of a closed 3-braid*, Journal of Knot Theory and its Ramifications **19** (2010), no. 9, 1157–1181.

[AK80] Selman Akbulut and Robion Kirby, *Branched covers of surfaces in 4-manifolds*, Math. Ann. **252** (1979/80), no. 2, 111–131.

[AP04] Marta Asaeda and Józef Przytycki, *Khovanov homology: torsion and thickness*, Advances in topological quantum field theory, no. 179 of NATO Scientific Series II: Mathematics, Physics and Chemistry, Kluwer Academic Publishers, Dordrecht, 2004.

[BGL] Kenneth Baker, Cameron McA. Gordon, and John Luecke, *Small Seifert fiber spaces from Dehn surgery*, In preparation.

[BGZ11] Steven Boyer, Cameron McA. Gordon, and Xingru Zhang, *Dehn fillings of knot manifolds containing essential once-punctured tori*, arXiv:1109.5151v3 [math.GT] (2011).

[BGZ12] ______, *Characteristic submanifold theory and toroidal Dehn filling*, Advances in Mathematics **230** (2012), no. 4-6, 1673–1737.

[BM70] Gerhard Burde and Kunio Murasugi, *Links and Seifert fiber spaces*, Duke Mathematics Journal **37** (1970), 89–93.

[Bri98] Mark Brittenham, *Exceptional Seifert-fibered spaces and Dehn surgery on 2-bridge knots*, Topology **37** (1998), no. 3, 665–672.

[BW01] Mark Brittenham and Ying-Qing Wu, *The classification of exceptional Dehn surgeries on 2-bridge knots*, Communications in Analysis and Geometry **9** (2001), no. 1, 97–113.

[CGLS87] Marc Culler, C. McA. Gordon, J. Luecke, and Peter B. Shalen, *Dehn surgery on knots*, Ann. of Math. (2) **125** (1987), no. 2, 237–300.

[Con70] John H. Conway, *An enumeration of knots and links, and some of their algebraic properties*, Computational Problems in Abstract Algebra (Pergamon, Oxford) (J. Leech, ed.), vol. Proc. Conf., Oxford, 1967, 1970, pp. 329–358.

[COT03] Tim D. Cochran, Kent E. Orr, and Peter Teichner, *Knot concordance, Whitney towers and L^2-signatures*, Ann. of Math. (2) **157** (2003), no. 2, 433–519.

[Cro04] Peter R. Cromwell, *Knots and links*, Cambridge University Press, Cambridge, 2004.

[Deh10] Max Dehn, *Über die Topologie des dreidimensionalen Raumes*, Mathematische Annalen **69** (1910), no. 1, 137–168.

[Del95] Charles Delman, *Essential laminations and Dehn surgery on 2-bridge knots*, Topology and its Applications **63** (1995), no. 3, 201–221.

[Don12] Andrew Donald, *Embedding Seifert manifolds in S^4*, arXiv:1203.6008, 2012.

[EM97] Mario Eudave-Muñoz, *Non-hyperbolic manifolds obtained by Dehn surgery on hyperbolic knots*, Geometric Topology, Studies in Advanced Mathematics (William H. Kazez, ed.), vol. 2.1, AMS and International Press, 1997, pp. 35–61.

[EM02] Mario Eudave-Muñoz, *On hyperbolic knots with Seifert fibered Dehn surgeries*, Proceedings of the First Joint Japan-Mexico Meeting in Topology (Morelia, 1999), vol. 121, 2002, pp. 119–141.

[EN85] David Eisenbud and Walter Neumann, *Three-dimensional link theory and invariants of plane curve singularities*, Annals of Mathematics Studies, 110, Princeton University Press, Princeton, NJ, 1985.

[FIK$^+$09] David Futer, Masaharu Ishikawa, Yuichi Kabaya, Thomas Mattman, and Koya Shimokawa, *Finite surgeries on three-tangle pretzel knots*, Algebraic & Geometric Topology **9** (2009), no. 2, 743–771.

[Fox62] R. H. Fox, *Some problems in knot theory*, Topology of 3-manifolds and related topics (Proc. The Univ. of Georgia Institute, 1961), Prentice-Hall, Englewood Cliffs, N.J., 1962, pp. 168–176.

[FQ90] Michael H. Freedman and Frank Quinn, *Topology of 4-manifolds*, Princeton Mathematical Series, vol. 39, Princeton University Press, Princeton, NJ, 1990.

[Fre82] Michael Hartley Freedman, *The topology of four-dimensional manifolds*, J. Differential Geom. **17** (1982), no. 3, 357–453.

[Fri04] Stefan Friedl, *Eta invariants as sliceness obstructions and their relation to Casson-Gordon invariants*, Algebr. Geom. Topol. **4** (2004), 893–934.

[Gab89] David Gabai, *Surgery on knots in solid tori*, Topology **28** (1989), no. 1, 1–6.

[GL83] Patrick M. Gilmer and Charles Livingston, *On embedding 3-manifolds in 4-space*, Topology **22** (1983), no. 3, 241–252.

[GL89] Cameron McA. Gordon and John Luecke, *Knots are determined by their complements*, Jounal of the American Mathematical Society **2** (1989), no. 2, 371–415.

[Gor99] Cameron McA. Gordon, *Small surfaces and Dehn filling*, Proceedings of the Kirbyfest (Berkeley, CA 1998) (Coventry), vol. 2, Geometry and Topology Monographs, Geometry and Topology Publishing, 1999, pp. 177–199.

[Gor09a] Cameron Gordon, *Dehn surgery and 3-manifolds*, Low dimensional topology, IAS/Park City Math. Ser., vol. 15, Amer. Math. Soc., Providence, RI, 2009, pp. 21–71.

[Gor09b] Cameron McA. Gordon, *Dehn surgery and 3-manifolds*, Low dimensional topology **IAS/Park City Mathematics Series, 15** (2009), 21–71.

[GRS08] J. Elisenda Grigsby, Daniel Ruberman, and Sašo Strle, *Knot concordance and Heegaard Floer homology invariants in branched covers*, Geom. Topol. **12** (2008), no. 4, 2249–2275.

[GS99] Robert E. Gompf and András I. Stipsicz, 4*-manifolds and Kirby calculus*, Graduate Studies in Mathematics, vol. 20, American Mathematical Society, Providence, RI, 1999.

[Han38] W. Hantzsche, *Einlagerung von Mannigfaltigkeiten in euklidische Räume*, Math. Z. **43** (1938), no. 1, 38–58.

[Hei74] Wolfgang Heil, *Elementary surgery on Seifert fiber spaces*, Yokohama Math Journal **22** (1974), 135–139.

[HKL12] Matthew Hedden, Se-Goo Kim, and Charles Livingston, *Topologically slice knots of smooth concordance order two*, arXiv:1212.6628, 2012.

[HLR12] Matthew Hedden, Charles Livingston, and Daniel Ruberman, *Topologically slice knots with nontrivial Alexander polynomial*, Adv. Math. **231** (2012), no. 2, 913–939.

[IJ09] Kazuhiro Ichihara and In Dae Jong, *Cyclic and finite surgeries on Montesinos knots*, Algebraic & Geometric Topology **9** (2009), no. 2, 731–742.

[IJ11] ______, *Seifert fibered surgery and Rasmussen invariant*, arXiv:1102.1117v1 [math.GT], 2011.

[IJK11] Kazuhiro Ichihara, In Dae Jong, and Yuichi Kabaya, *Exceptional surgeries on* $(-2, p, p)$*-pretzel knots*, arXiv:1102.1118v1 [math.GT], 2011.

[IM13] Kazuhiro Ichihara and Hidetoshi Masai, *Exceptional surgeries on alternating knots*, arXiv:1310.3472, 2013.

[JN07] Stanislav Jabuka and Swatee Naik, *Order in the concordance group and Heegaard Floer homology*, Geom. Topol. **11** (2007), 979–994.

[Kan10] Sungmo Kang, *Examples of reducible and finite Dehn fillings*, Journal of Knot Theory and its Ramifications **19** (2010), no. 5, 677–694.

[Ker71] Michel A. Kervaire, *Knot cobordism in codimension two*, Manifolds–Amsterdam 1970 (Proc. Nuffic Summer School), Lecture Notes in Mathematics, Vol. 197, Springer, Berlin, 1971, pp. 83–105.

[Kim06] Taehee Kim, *New obstructions to doubly slicing knots*, Topology **45** (2006), no. 3, 543–566.

[Lee11] Sangyop Lee, *Lens spaces and toroidal Dehn fillings*, Mathematische Zeitschrift **267** (2011), no. 3-4, 781–802.

[Lev69a] J. Levine, *Invariants of knot cobordism*, Invent. Math. 8 (1969), 98–110; addendum, ibid. **8** (1969), 355.

[Lev69b] ______, *Knot cobordism groups in codimension two*, Comment. Math. Helv. **44** (1969), 229–244.

[Lev89] J. P. Levine, *Metabolic and hyperbolic forms from knot theory*, J. Pure Appl. Algebra **58** (1989), no. 3, 251–260.

[Lic97] W. B. Raymond Lickorish, *An Introduction to Knot Theory*, Graduate Texts in Mathematics, 175, Springer-Verlag, New York, 1997.

[Liv05] Charles Livingston, *A survey of classical knot concordance*, Handbook of knot theory, Elsevier B. V., Amsterdam, 2005, pp. 319–347.

[LM08] Marc Lackenby and Robert Meyerhoff, *The maximal number of exceptional Dehn surgeries*, arXiv:0808.1176v1 [math.GT] (2008).

[LS04] Paolo Lisca and András I. Stipsicz, *Ozsváth-Szabó invariants and tight contact three-manifolds. I*, Geom. Topol. **8** (2004), 925–945 (electronic).

[LT88] W. B. Raymond Lickorish and Morwen B. Thistlethwaite, *Some links with nontrivial polynomials and their crossing-numbers*, Commentarii Mathematici Helvetici **63** (1988), no. 4, 527–539.

[MB84] John W. Morgan and Hyman Bass (eds.), *The Smith Conjecture*, Pure and Applied Mathematics, 112, Academic Press, Orlando, FL, 1984.

[Mei13] Jeffrey Meier, *Distinguishing topologically and smoothly doubly slice knots*, in preparation, 2013.

[Mei14] ______, *Small Seifert fibered surgery on hyperbolic pretzel knots*, Algebr. Geom. Topol. **14** (2014), no. 1, 439–487.

[MM02] Katura Miyazaki and Kimihiko Motegi, *Seifert fibering surgery on periodic knots*, Topology and its Applications **121** (2002), no. 1-2, 275–285.

[MMM06] Thomas Mattman, Katura Miyazaki, and Kimihiko Motegi, *Seifert-fibered surgeries which do not arise from primitive/Seifert-fibered constructions*, Trans. Amer. Math. Soc. **358** (2006), no. 9, 4045–4055.

[Mon75] José M. Montesinos, *Surgery on links and double branched covers of S^3*, Knots, groups, and 3-manifolds (Papers dedicated to the memory of R. H. Fox, vol. 84, Princeton University Press, 1975, pp. 227–259.

[Mot03] Kimihiko Motegi, *Dehn surgeries, group actions, and Seifert fiber spaces*, Communications in Analysis and Geometry **11** (2003), no. 2, 343–389.

[MSY96] Kanji Morimoto, Makoto Sakuma, and Yoshiyuki Yokota, *Identifying tunnel number one knots*, Journal of the Mathematical Society of Japan **48** (1996), no. 4, 667–688.

[NW10] Yi Ni and Zhongtao Wu, *Cosmetic surgeries on knots in S^3*, arXiv:1009.4720v2, 2010.

[NW12] ———, *Heegaard Floer correction terms and rational genus bounds*, arXiv:1205.7053, 2012.

[Oer84] Ulrich Oertel, *Closed incompressible surfaces in complements of star links*, Pacific Journal of Mathematics **111** (1984), no. 1, 209–230.

[OS03a] Peter Ozsváth and Zoltán Szabó, *Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary*, Adv. Math. **173** (2003), no. 2, 179–261.

[OS03b] ———, *Heegaard Floer homology and alternating knots*, Geom. Topol. **7** (2003), 225–254 (electronic).

[OS04a] ———, *Holomorphic disks and knot invariants*, Adv. Math. **186** (2004), no. 1, 58–116.

[OS04b] ———, *Holomorphic disks and topological invariants for closed three-manifolds*, Ann. of Math. (2) **159** (2004), no. 3, 1027–1158.

[OS06] ———, *Lectures on Heegaard Floer homology*, Floer homology, gauge theory, and low-dimensional topology, Clay Math. Proc., vol. 5, Amer. Math. Soc., Providence, RI, 2006, pp. 29–70.

[Per02] Grisha Perelman, *The entropy formula for Ricci flow and its geometric applications*, arXiv:math.DG/0211159, 2002.

[Per03a] ———, *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds*, arXiv:math.DG/0307245, 2003.

[Per03b] ———, *Ricci flow with surgery on three-manifolds*, arXiv:math.DG/0303109, 2003.

[Ras10] Jacob Rasmussen, *Khovanov homology and the slice genus*, Inventiones Mathematicae **182** (2010), no. 2, 419–447.

[Sum71] D. W. Sumners, *Invertible knot cobordisms*, Comment. Math. Helv. **46** (1971), 240–256.

[Thu82a] W. Thurston, *Hyperbolic geometry and* 3*-manifolds*, Low-dimensional topology (Bangor, 1979), London Math. Soc. Lecture Note Ser., vol. 48, Cambridge Univ. Press, Cambridge, 1982, pp. 9–25.

[Thu82b] William P. Thurston, *Three-dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Amer. Math. Soc. (N.S.) **6** (1982), no. 3, 357–381.

[Tur06] Paul Turner, *Five lectures on Khovanov homology*, arXiv:math/0606464 [math.GT], 2006.

[Wal63] C. T. C. Wall, *Quadratic forms on finite groups, and related topics*, Topology **2** (1963), 281–298.

[Wol99] Stephan Wolfram, *The Mathematica® Book*, fourth ed., Wolfram Media, Inc., Champaign, IL; Cambridge University Press, Cambridge, 1999.

[Wu96] Ying-Qing Wu, *Dehn surgery on arborescent knots*, Journal of Differential Geometry **43** (1996), no. 1, 171–197.

[Wu98a] ———, *Dehn surgery on arborescent knots and links-a survey*, Chaos Solitons Fractals **9** (1998), no. 4-5, 671–679.

[Wu98b] ———, *Sutured manifold hierarchies, essential laminations, and Dehn surgery*, Journal of Differential Geometry **48** (1998), no. 3, 407–437.

[Wu10] ———, *Persistently laminar branched surfaces*, arXiv:1008.2680v1 [math.GT], 2010.

[Wu11a] ———, *The classification of toroidal Dehn surgeries on Montesinos knots*, Communications in Analysis and Geometry **19** (2011), no. 2, 305–345.

[Wu11b] ———, *Exceptional Dehn surgery on large arborescent knots*, Pacific Journal of Mathematics **252** (2011), no. 1, 219–243.

[Wu11c] ———, *Immersed surfaces and Seifert fibered surgery on Montesinos knots*, arXiv:0910.4882v2 [math.GT], 2011.

[Wu12a] ———, *Dehn surgery on knots of wrapping nuber 2*, arXiv:1105.4287v1, 2012.

[Wu12b] ———, *Seifert fibered surgery on Montesinos knots*, arXiv:1207.0154, 2012.

[Wu12c] ———, *Seifert fibered surgery on Montesinos knots*, arXiv:12007.0154v1, 2012.

[Wu13] Ying-Qing Wu, *Dehn surgery on knots of wrapping number 2*, Algebr. Geom. Topol. **13** (2013), no. 1, 479–503.

[Zee65] E. C. Zeeman, *Twisting spun knots*, Trans. Amer. Math. Soc. **115** (1965), 471–495.